

# Outage Probability and Outage-Based Robust Beamforming for MIMO Interference Channels with Imperfect Channel State Information

Juho Park, *Student Member, IEEE*, Youngchul Sung<sup>†</sup>, *Senior Member, IEEE*, Donggun Kim,  
*Student Member, IEEE*, and H. V. Poor *Fellow, IEEE*

## Abstract

In this paper, the outage probability and outage-based beam design for multiple-input multiple-output (MIMO) interference channels are considered. First, closed-form expressions for the outage probability in MIMO interference channels are derived under the assumption of Gaussian-distributed channel state information (CSI) error, and the asymptotic behavior of the outage probability as a function of several system parameters is examined by using the Chernoff bound. It is shown that the outage probability decreases exponentially with respect to the quality of CSI measured by the inverse of the mean square error of CSI. Second, based on the derived outage probability expressions, an iterative beam design algorithm for maximizing the sum outage rate is proposed. Numerical results show that the proposed beam design algorithm yields better sum outage rate performance than conventional algorithms such as interference alignment developed under the assumption of perfect CSI.

## Index Terms

Multiuser MIMO, interference channels, channel uncertainty, outage probability, Chernoff bound, interference alignment

<sup>†</sup>Corresponding author

J. Park, Y. Sung and D. Kim are with the Dept. of Electrical Engineering, KAIST, Daejeon 305-701, South Korea. E-mail: {jhp@, ysung@ee. and dg.kim@}kaist.ac.kr and H. V. Poor is with Dept. of Electrical Engineering, Princeton University, Princeton, NJ 08544, E-mail: poor@princeton.edu This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-220-D00079). A preliminary version of this work is presented at IEEE Globecom 2011. [1].

## I. INTRODUCTION

Due to their importance in current and future wireless communication systems, multiple-input multiple-output (MIMO) interference channels have gained much attention from the research community in recent years. Since Cadambe and Jafar showed that interference alignment (IA) achieved the maximum number of degrees of freedom in MIMO interference channels [2], there has been extensive research in devising good beam design algorithms for MIMO interference channels. Now, there are many available beam design algorithms for MIMO interference channels such as IA-based algorithms [3]–[5] and sum-rate targeted algorithms [3], [4], [6]–[9]. However, most of these algorithms assume perfect channel state information (CSI) at transmitters and receivers, whereas the assumption of perfect CSI is unrealistic in practical wireless communication systems since perfect CSI is unavailable in practical wireless communication systems due to channel estimation error, limited feedback or other limitations [10]. Thus, the CSI error should be incorporated into the beam design to yield better performance, and this is typically done under robust beam design frameworks.

There are many robust beam design studies in the conventional single-user MIMO case and also in the multiple-input and single-output (MISO) multi-user case. In the MISO multi-user case, the problem is more tractable than in the MIMO multi-user case, and extensive research results are available on MISO broadcast and interference channels with imperfect CSI [11]–[13]; the outage rate region is defined for MISO interference channels in [11], and the optimal beam structure that achieves a Pareto-optimal point of the outage rate region is given in [12]. For more complicated MIMO interference channels, there are several pioneering works on robust beam design under CSI uncertainty [14]–[16]. In [14], the authors solved the problem based on a worst-case approach. In their work, the CSI error is modelled as a random variable under a Frobenius norm constraint, and a semi-definite relaxation method is used to obtain the beam vectors that maximize the minimum signal-to-interference-plus-noise ratio (SINR) over all users and all possible CSI error. In [15], on the other hand, the CSI error is modelled as an independent Gaussian random variable, and the beam is designed to minimize the mean square error (MSE) between the transmitted signal and the reconstructed signal at the receiver with given imperfect CSI at the transmitter (CSIT).

In this paper, we consider the robust beam design in MIMO interference channels based on a different criterion. Here, we consider the rate outage due to channel uncertainty and the problem of sum rate maximization under an outage constraint in MIMO interference channels. This formulation is practically meaningful since an outage probability is assigned to each user and the supportable rate with the given

outage probability is maximized in practical systems. Here, we assume that the transmitters and receivers have imperfect CSI and the CSI error is circularly-symmetric complex Gaussian distributed. Under this assumption, we first derive closed-form expressions for the outage probability in MIMO interference channels for an arbitrarily given set of transmit and receive beamforming vectors, and then derive the asymptotic behavior of the outage probability as a function of several system parameters by using the Chernoff bound. It is shown that *the outage probability decreases exponentially with respect to (w.r.t.) the quality of CSI measured by the inverse of the MSE of CSI, typically called the channel K factor [10] or interpreted as the Fisher information [17] in statistical estimation theory.* In particular, it is shown that in the case of interference alignment, the outage probability can be made arbitrarily small by improving the CSI quality if the target rate is strictly less than the rate obtained by using the estimated as the nominal channel. Next, based on the derived outage probability expressions, we propose an iterative beam design algorithm for maximizing the weighted sum rate under the constraint that the outage probability for each user is less than a certain level. Numerical results show that the proposed beam design algorithm yields better sum outage rate performance than conventional beam design algorithms such as the ‘max-SINR’ algorithm [3] developed without the consideration of channel uncertainty.

#### A. Related work

The outage analysis for MIMO interference channels has been performed by several other researchers [16], [18]. In [16], the outage probability for a given rate tuple is computed under the assumption that the knowledge of the channel mean and covariance matrix are available, and transmit and receive beam vectors that minimize the power consumption for a given outage constraint are obtained. However, it is difficult to generalize this method of analysis to the case of multiple data streams per user, whereas our analysis includes the multiple data stream case. In [18], the outage probability and SINR distribution of each user in MIMO interference channels with the knowledge of channel distribution information are obtained under a particular transmit and receive beam structure of IA transmit beams and zero-forcing (ZF) receivers. On the other hand, our analysis can be applied to the case of general transmit and receive beam structures beyond IA and ZF.

The probability distribution of a quadratic form of Gaussian random variables has been studied extensively in the statistics field [19]–[22] and in the communications area [23]–[25]. The most widely-used approach to obtain the probability distribution of a Gaussian quadratic form is the series fitting method [20], [21], [23], [26], which typically converges to the probability distribution of a Gaussian quadratic form from the lower tail first. However, the outage definition associated with robust beam design

for MIMO interference channels in this paper requires accurate computation of upper tail probabilities. The series expansion for the cumulative distribution function (CDF) obtained in this paper based on the integral form for the CDF in [25] and the residue theorem [22] is well suited to this purpose and converges to the upper tail first. Thus, the obtained series in this paper is more relevant for our outage analysis. For a detailed explanation of the derived series, please see Appendices B–C.

### B. Notation and organization

We will make use of standard notational conventions. Vectors and matrices are written in boldface with matrices in capitals. All vectors are column vectors. For a matrix  $\mathbf{A}$ ,  $\mathbf{A}^H$ ,  $\|\mathbf{A}\|_F$  and  $\mathbf{A}(i, j)$  indicate the Hermitian transpose, the Frobenius norm and the element in row  $i$  and column  $j$  of  $\mathbf{A}$ , respectively, and  $\text{vec}(\mathbf{A})$  and  $\text{tr}(\mathbf{A})$  denote the vector composed of the columns of  $\mathbf{A}$  and the trace of  $\mathbf{A}$ , respectively. For vector  $\mathbf{a}$ ,  $\|\mathbf{a}\|$  and  $[\mathbf{a}]_i$  represent the 2-norm and the  $i$ -th element of  $\mathbf{a}$ , respectively.  $\mathbf{I}_n$  stands for the identity matrix of size  $n$  (the subscript is included only when necessary), and  $\text{diag}(d_1, \dots, d_n)$  means a diagonal matrix with diagonal elements  $d_1, \dots, d_n$ .  $\mathbf{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  means that the random vector  $\mathbf{x}$  has the circularly-symmetric complex Gaussian distribution with mean vector  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$ .  $\mathcal{K} = \{1, 2, \dots, K\}$ ,  $\iota = \sqrt{-1}$ , and  $|A|$  denotes the cardinality of the set  $A$ .

The paper is organized as follows. The system model and problem formulation are described in Section II. In Section III, closed-form expressions for the outage probability are derived, and the behavior of the outage probability as a function of several system parameters is examined by using the Chernoff bound. In Section IV, an outage-based beam design algorithm is proposed. Numerical results are provided in Section V, followed by the conclusion in Section VI.

## II. SYSTEM MODEL AND PROBLEM FORMULATION

In this paper, we consider a  $K$ -user time-invariant MIMO interference channel in which each transmitter equipped with  $N_t$  antennas is paired with a receiver equipped with  $N_r$  antennas, and interferes with all receivers other than the desired receiver. We assume that transmitter  $k$  transmits  $d$  ( $\leq \min(N_t, N_r)$ ) independent data streams to receiver  $k$  paired with transmitter  $k$ . Then, the received signal at receiver  $k$  is given by

$$\mathbf{y}_k = \mathbf{H}_{kk} \mathbf{V}_k \mathbf{s}_k + \sum_{i=1, i \neq k}^K \mathbf{H}_{ki} \mathbf{V}_i \mathbf{s}_i + \mathbf{n}_k, \quad (1)$$

where  $\mathbf{H}_{ki}$  is the  $N_r \times N_t$  channel matrix from transmitter  $i$  to receiver  $k$ ;  $\mathbf{V}_i = [\mathbf{v}_i^{(1)}, \dots, \mathbf{v}_i^{(d)}]$  is the  $N_t \times d$  transmit beamforming matrix with normalized column vectors at transmitter  $i$ , i.e.,  $\|\mathbf{v}_i^{(m)}\| = 1$

for  $m = 1, \dots, d$ ; and  $\mathbf{s}_i = [s_i^{(1)}, \dots, s_i^{(d)}]^T$  is the  $d \times 1$  symbol vector at transmitter  $i$ . We assume that the transmit symbol vector  $\mathbf{s}_i$  is drawn from the zero-mean Gaussian distribution with unit variance, i.e.,  $\mathbf{s}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ , and the additive noise vector  $\mathbf{n}_k$  is zero-mean Gaussian distributed with variance  $\sigma^2$ , i.e.,  $\mathbf{n}_k \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I})$ . We assume that the CSI available to the system is not perfect. That is, neither the transmitters nor the receivers have perfect CSI. For the imperfect CSI, we adopt the following model

$$\mathbf{H}_{ki} = \hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki} \quad (2)$$

for each  $(k, i) \in \mathcal{K} \times \mathcal{K}$ , where  $\mathbf{H}_{ki}$  is the unknown true channel,  $\hat{\mathbf{H}}_{ki}$  is the channel state available to the transmitters and the receivers, and  $\mathbf{E}_{ki}$  is the error between the true and available channel information. For the CSI error  $\mathbf{E}_{ki}$  between the true and available channel information, we adopt the Kronecker error model which is widely used for MIMO systems to model the error correlation that may be caused by the transmit and receive antenna structure [10]. Under this model, the CSI error  $\mathbf{E}_{ki}$  is given by

$$\mathbf{E}_{ki} = \Sigma_r^{1/2} \mathbf{H}_{ki}^{(w)} \Sigma_t^{1/2}, \quad \text{with } \text{vec}(\mathbf{H}_{ki}^{(w)}) \sim \mathcal{CN}(\mathbf{0}, \sigma_h^2 \mathbf{I}) \quad \text{for some } \sigma_h^2 \geq 0, \quad (3)$$

where  $\Sigma_t$  and  $\Sigma_r$  are transmit and receive antenna correlation matrices, respectively, and the elements of  $\mathbf{H}_{ki}^{(w)}$  are independent and identically distributed (i.i.d.) and are drawn from a circularly-symmetric zero-mean complex Gaussian distribution. The CSI uncertainty matrix  $\mathbf{E}_{ki}$  is a circularly-symmetric<sup>1</sup> complex Gaussian random matrix with distribution  $\text{vec}(\mathbf{E}_{ki}) \sim \mathcal{CN}(\mathbf{0}, \sigma_h^2 (\Sigma_t^T \otimes \Sigma_r))$  [10, p.90], and  $\sigma_h^2$  is the parameter capturing the uncertainty level in CSI. We assume that the  $\mathbf{E}_{ki}$ 's are independent across transmitter-receiver pairs  $(k, i)$ . To specify the quality of CSI and signal reception, we define two parameters

$$K_{ch}^{(ki)} := \frac{\|\hat{\mathbf{H}}_{ki}\|_F^2}{\mathbb{E}\{\|\mathbf{E}_{ki}\|_F^2\}} = \frac{\|\hat{\mathbf{H}}_{ki}\|_F^2}{\sigma_h^2 \text{tr}(\Sigma_t^T \otimes \Sigma_r)} \quad \text{and} \quad \Gamma^{(k)} := \frac{\|\hat{\mathbf{H}}_{kk}\|_F^2}{\sigma^2}.$$

$K_{ch}^{(ki)}$  is the channel  $K$  factor defined as the ratio of the power of the known channel part to that of the unknown channel part, representing the quality of CSI [10], and  $\Gamma^{(k)}$  is the signal-to-noise ratio (SNR) at receiver  $k$  since  $\mathbf{V}_k$  and  $\mathbf{s}_k$  are normalized in our formulation. Hereafter, we will use  $\hat{\mathcal{H}}$  to represent the collection of channel information  $\{\hat{\mathbf{H}}_{ki}, \Sigma_t, \Sigma_r\}$  known to the transmitters and receivers. By using the receiver filter  $\mathbf{u}_k^{(m)}$  ( $\|\mathbf{u}_k^{(m)}\| = 1$ ), receiver  $k$  projects the received signal  $\mathbf{y}_k$  in (1) to recover the desired signal stream  $m$ :

$$\hat{s}_k^{(m)} = (\mathbf{u}_k^{(m)})^H \mathbf{y}_k = (\mathbf{u}_k^{(m)})^H \left( (\hat{\mathbf{H}}_{kk} + \mathbf{E}_{kk}) \mathbf{V}_k \mathbf{s}_k + \sum_{i=1, i \neq k}^K (\hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki}) \mathbf{V}_i \mathbf{s}_i + \mathbf{n}_k \right).$$

<sup>1</sup>The circular symmetry of a random matrix in form of  $\mathbf{AZB}$  with constant matrices  $\mathbf{A}$  and  $\mathbf{B}$  and a circularly-symmetric complex Gaussian matrix  $\mathbf{Z}$  can easily be shown by a similar technique to that used in the Appendix A.

We assume that the design of the transmit beamforming matrices  $\{\mathbf{V}_k, k \in \mathcal{K}\}$  and receive filters  $\{\mathbf{U}_k = [\mathbf{u}_k^{(1)}, \dots, \mathbf{u}_k^{(d)}], k \in \mathcal{K}\}$  is based on the available CSI  $\hat{\mathcal{H}}$ . This model of beam design and signal transmission and reception captures many coherent linear beamforming MIMO schemes including interference alignment and sum rate maximizing beamforming schemes [3], [6], [27] in which transmit and receive beamforming matrices are designed based on available CSI at transmitters and receivers. Under this processing model, the SINR for stream  $m$  of user  $k$  is given by

$$\text{SINR}_k^{(m)} \big|_{\hat{\mathcal{H}}} = \frac{|(\mathbf{u}_k^{(m)})^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k^{(m)}|^2}{|(\mathbf{u}_k^{(m)})^H \mathbf{E}_{kk} \mathbf{v}_k^{(m)}|^2 + \sum_{j \neq m} |(\mathbf{u}_k^{(m)})^H (\hat{\mathbf{H}}_{kk} + \mathbf{E}_{kk}) \mathbf{v}_k^{(j)}|^2 + \sum_{i \neq k} \sum_{j=1}^d |(\mathbf{u}_k^{(m)})^H (\hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki}) \mathbf{v}_i^{(j)}|^2 + \sigma^2}, \quad (4)$$

where the numerator of the right-hand side (RHS) in (4) is the desired signal power, and the first, second, third and fourth terms in the denominator of the RHS in (4) represent the interference purely by channel uncertainty, inter-stream interference, other user interference and thermal noise, respectively. (Here, the dependence of SINR on  $\hat{\mathcal{H}}$  is explicitly shown. Since the dependence is clear, the notation  $|_{\hat{\mathcal{H}}}$  will be omitted hereafter.) Because the  $\{\mathbf{E}_{ki}\}$  are random,  $\text{SINR}_k^{(m)}$  is a random variable for given  $\hat{\mathcal{H}}$  and  $\{\mathbf{V}_k(\hat{\mathcal{H}}), \mathbf{U}_k(\hat{\mathcal{H}}), k \in \mathcal{K}\}$ . Thus, an outage at stream  $m$  of user  $k$  occurs if the supportable rate determined by the received SINR (4) is below the target rate  $R_k^{(m)}$ , and the outage probability is given by

$$\Pr\{\text{outage}\} = \Pr\left\{\log_2\left(1 + \text{SINR}_k^{(m)}\right) \leq R_k^{(m)}\right\}. \quad (5)$$

By rearranging the terms in (4), the outage event can be expressed as

$$\sum_{i=1}^K \sum_{j=1}^d X_{ki}^{(mj)H} X_{ki}^{(mj)} \geq \frac{|\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{kk} \mathbf{v}_k^{(m)}|^2}{2^{R_k^{(m)}} - 1} - \sigma^2 =: \tau, \quad (6)$$

where

$$X_{ki}^{(mj)} := \begin{cases} \mathbf{u}_k^{(m)H} \mathbf{E}_{kk} \mathbf{v}_k^{(m)}, & i = k \text{ and } j = m, \\ \mathbf{u}_k^{(m)H} (\hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki}) \mathbf{v}_i^{(j)}, & \text{otherwise.} \end{cases} \quad (7)$$

Since the  $\{\mathbf{E}_{ki}\}$  are circularly-symmetric complex Gaussian random matrices,  $\{X_{ki}^{(mj)}, i = 1, \dots, K, j = 1, \dots, d\}$  are circularly-symmetric complex Gaussian random variables, and the left-hand side (LHS) of (6) is a *quadratic form of non-central Gaussian random variables*. To simplify notation, we will use vector form from here on. In vector form, (6) can be expressed as

$$\mathbf{X}_k^{(m)H} \mathbf{X}_k^{(m)} \geq \tau, \quad (8)$$

where  $\mathbf{X}_k^{(m)} := [X_{k1}^{(m1)}, \dots, X_{k1}^{(md)}, X_{k2}^{(m1)}, \dots, X_{kK}^{(md)}]^T$ . The elements of the mean vector  $\boldsymbol{\mu}_k^{(m)} (:= \mathbb{E}\{\mathbf{X}_k^{(m)}\})$  of  $\mathbf{X}_k^{(m)}$  are given by

$$[\boldsymbol{\mu}_k^{(m)}]_{(i-1)d+j} = \begin{cases} 0, & i = k, j = m, \\ \mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{ki} \mathbf{v}_i^{(j)}, & \text{otherwise,} \end{cases} \quad (9)$$

for  $i = 1, \dots, K$  and  $j = 1, \dots, d$ , and the covariance matrix  $\boldsymbol{\Sigma}_k^{(m)}$  of  $\mathbf{X}_k^{(m)}$  is given by a block diagonal matrix, since  $\{\mathbf{E}_{ki}, i = 1, \dots, K\}$  are independent for different values of  $i$ , i.e.,

$$\boldsymbol{\Sigma}_k^{(m)} := \mathbb{E}\{(\mathbf{X}_k^{(m)} - \mathbb{E}\{\mathbf{X}_k^{(m)}\})(\mathbf{X}_k^{(m)} - \mathbb{E}\{\mathbf{X}_k^{(m)}\})^H\} = \text{diag}(\boldsymbol{\Sigma}_{k,1}^{(m)}, \dots, \boldsymbol{\Sigma}_{k,K}^{(m)}), \quad (10)$$

where the  $d \times d$  sub-block matrix  $\boldsymbol{\Sigma}_{k,i}^{(m)}$  is given by

$$\boldsymbol{\Sigma}_{k,i}^{(m)} = \sigma_h^2 (\mathbf{u}_k^{(m)H} \boldsymbol{\Sigma}_r \mathbf{u}_k^{(m)}) \begin{bmatrix} \mathbf{v}_i^{(1)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(1)} & \mathbf{v}_i^{(2)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(1)} & \dots & \mathbf{v}_i^{(d)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(1)} \\ \mathbf{v}_i^{(1)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(2)} & \mathbf{v}_i^{(2)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(2)} & \dots & \mathbf{v}_i^{(d)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_i^{(1)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(d)} & \mathbf{v}_i^{(2)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(d)} & \dots & \mathbf{v}_i^{(d)H} \boldsymbol{\Sigma}_t \mathbf{v}_i^{(d)} \end{bmatrix} \quad (11)$$

for each  $i$ . (The proof of (11) is given in Appendix A.) In the following sections, we will derive closed-form expressions for (5), investigate the behavior of the outage probability as a function of several parameters, and propose an outage-based beam design algorithm.

### III. THE COMPUTATION OF THE OUTAGE PROBABILITY

In this section, we first derive a closed-form expression for the outage probability in the general case of the Kronecker CSI error model, and then consider special cases. After this, we examine the behavior of the outage probability as a function of several important system parameters based on the Chernoff bound.

#### A. Closed-form expressions for the outage probability

For a Gaussian random vector  $\mathbf{X} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  with the eigendecomposition of its covariance matrix  $\boldsymbol{\Sigma} = \boldsymbol{\Psi} \boldsymbol{\Lambda} \boldsymbol{\Psi}^H$ , the CDF of  $\mathbf{X}^H \bar{\mathbf{Q}} \mathbf{X}$  for some given  $\bar{\mathbf{Q}}$  is given by [25]

$$\Pr\{\mathbf{X}^H \bar{\mathbf{Q}} \mathbf{X} \leq \tau\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\tau(\omega + \beta)}}{\omega + \beta} \frac{e^{-c}}{\det(\mathbf{I} + (\omega + \beta)\mathbf{Q})} d\omega \quad (12)$$

for some  $\beta > 0$  such that  $\mathbf{I} + \beta\mathbf{Q}$  is positive definite, where  $\mathbf{Q} = \boldsymbol{\Lambda}^{H/2} \boldsymbol{\Psi}^H \bar{\mathbf{Q}} \boldsymbol{\Psi} \boldsymbol{\Lambda}^{1/2}$ ,  $\boldsymbol{\chi} = \boldsymbol{\Lambda}^{-1/2} \boldsymbol{\Psi}^H \boldsymbol{\mu}$  and  $c = \boldsymbol{\chi}^H \left( \mathbf{I} + \frac{1}{\omega + \beta} \mathbf{Q} \right)^{-1} \boldsymbol{\chi}$ . From here on, we will derive closed-form series expressions for the CDF of the outage probability in several important cases by applying the residue theorem used in [22]



to the integral form (12) for the CDF. First, we consider the most general case of the Kronecker CSI error model. The outage probability in this case is given by the following theorem.

*Theorem 1:* For given transmit and receive beamforming matrices  $\{\mathbf{V}_k = [\mathbf{v}_k^{(1)}, \dots, \mathbf{v}_k^{(d)}]\}$  and  $\{\mathbf{U}_k = [\mathbf{u}_k^{(1)}, \dots, \mathbf{u}_k^{(d)}]\}$  designed based on  $\hat{\mathcal{H}} = \{\hat{\mathbf{H}}_{ki}, \Sigma_t, \Sigma_r\}$ , the outage probability for stream  $m$  of user  $k$  with the target rate  $R_k^{(m)}$  under the CSI error model (2) and (3) is given by

$$\begin{aligned} \Pr\{\text{outage}\} &= \Pr\{\log_2(1 + \text{SINR}_k^{(m)}) \leq R_k^{(m)}\} \\ &= -\sum_{i=1}^{\kappa} \frac{e^{-(\frac{\tau}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0) \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1} \end{aligned} \quad (13)$$

where  $\tau$  is given in (6);  $\{\lambda_i, i = 1, \dots, \kappa\}$  are all the distinct eigenvalues of the  $Kd \times Kd$  covariance matrix  $\Sigma_k^{(m)}$  in (10) with eigendecomposition  $\Sigma_k^{(m)} = \Psi_k^{(m)} \Lambda_k^{(m)} \Psi_k^{(m)H}$ ;  $\kappa_i$  is the multiplicity<sup>2</sup> of the eigenvalue  $\lambda_i$ ;  $\chi_i^{(j)}$  is the element of vector

$$\chi_k^{(m)} := (\Lambda_k^{(m)})^{-\frac{1}{2}} \Psi_k^{(m)H} \mu_k^{(m)} \quad (14)$$

corresponding to the  $j$ -th eigenvector of the eigenvalue  $\lambda_i$  ( $1 \leq j \leq \kappa_i$ ), i.e., it is the  $j$ -th element of  $(\lambda_i \mathbf{I}_{\kappa_i})^{-\frac{1}{2}} \Psi_{k,i}^{(m)H} \mu_k^{(m)}$ .  $(\Psi_{k,i}^{(m)})$  is a  $Kd \times \kappa_i$  matrix composed of the eigenvectors of  $\Sigma_k^{(m)}$  associated with  $\lambda_i$ ;

$$g_i(s) = \frac{e^{\tau s}}{s - 1/\lambda_i} \cdot \frac{\exp\left(-\sum_{p \neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2\right)}{\prod_{p \neq i} \left(1 + (s - 1/\lambda_i)\lambda_p\right)^{\kappa_p}}; \quad (15)$$

and  $g_i^{(n)}(s)$  is the  $n$ -th derivative of  $g_i(s)$  w.r.t.  $s$ .

*Proof:* By using (12) and the facts  $\bar{\mathbf{Q}} = \mathbf{I}$  and  $\mathbf{X}_k^{(m)} \sim \mathcal{CN}(\mu_k^{(m)}, \Sigma_k^{(m)})$  in this case, we obtain the outage probability for stream  $m$  of user  $k$  in an integral form as

$$\Pr\{\mathbf{X}_k^{(m)H} \mathbf{X}_k^{(m)} \geq \tau\} = 1 - \frac{1}{2\pi\iota} \int_{\beta - \iota\infty}^{\beta + \iota\infty} \frac{e^{s\tau}}{s} \cdot \frac{e^{-\sum_{i=1}^{\kappa} \frac{s\lambda_i}{1+s\lambda_i} (\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\prod_{i=1}^{\kappa} (1 + s\lambda_i)^{\kappa_i}} ds, \quad (16)$$

where  $s = \beta + \iota\omega$  ( $\beta > 0$ ). The outage probability (16) can be expressed as a contour integral:

$$\Pr\{\mathbf{X}_k^{(m)H} \mathbf{X}_k^{(m)} \geq \tau\} = 1 - \frac{1}{2\pi\iota} \oint_{\mathcal{C}} \underbrace{\frac{e^{s\tau}}{s} \cdot \frac{e^{-\sum_{i=1}^{\kappa} \frac{s\lambda_i}{1+s\lambda_i} (\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\prod_{i=1}^{\kappa} (1 + s\lambda_i)^{\kappa_i}}}_{=: F(s)} ds, \quad (17)$$

where  $\mathcal{C}$  is a contour of integration containing the imaginary axis and the whole left half plane of the complex plane. By the residue theorem, the sum of the residues at singular points of  $F(s)$  which do not have positive real parts yields the contour integral in (17) times  $2\pi\iota$ . It is easy to see that the singular

<sup>2</sup>Since  $\Sigma_k^{(m)}$  is a normal matrix, we have  $Kd = \sum_{i=1}^{\kappa} \kappa_i$ .



points of  $F(s)$  are  $s = 0$  and  $s = -1/\lambda_i$ ,  $i = 1, \dots, \kappa$ . Since  $\Sigma_{k,i}^{(m)}$  are all positive-definite,  $\Sigma_k^{(m)}$  is positive definite and  $\lambda_i > 0$  for all  $i$ . So, the outage probability is given by

$$\Pr\{\text{outage}\} = 1 - \left( \text{Res}_{s=0} F(s) + \sum_{i=1}^{\kappa} \text{Res}_{s=-1/\lambda_i} F(s) \right). \quad (18)$$

It is also easy to see from (17) that the residue of  $F(s)$  at  $s = 0$  is  $\text{Res}_{s=0} F(s) = 1$ . To compute  $\text{Res}_{s=-1/\lambda_i} F(s)$ , for each  $i$  we introduce  $G_i(s)$  defined as

$$\begin{aligned} G_i(s) &:= F\left(s - \frac{1}{\lambda_i}\right) = \frac{e^{\tau(s-1/\lambda_i)}}{s - 1/\lambda_i} \cdot \frac{e^{-\sum_{p=1}^{\kappa} \frac{\lambda_p(s-1/\lambda_i)}{1+\lambda_p(s-1/\lambda_i)} (\sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2)}}{\prod_{p=1}^{\kappa} (1 + \lambda_p(s - 1/\lambda_i))^{\kappa_p}} \\ &= \frac{e^{\tau(s-1/\lambda_i)}}{s - 1/\lambda_i} \cdot \frac{e^{-\frac{\lambda_i s - 1}{\lambda_i s} \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}}{(\lambda_i s)^{\kappa_i}} \cdot \underbrace{\frac{e^{-\sum_{p \neq i} \frac{\lambda_p(s-1/\lambda_i)}{1+\lambda_p(s-1/\lambda_i)} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2}}{\prod_{p \neq i} (1 + \lambda_p(s - 1/\lambda_i))^{\kappa_p}}}_{=: I_1} \\ &= e^{-(\frac{\tau}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)} \times \underbrace{\frac{e^{\frac{1}{\lambda_i s} \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}}{(\lambda_i s)^{\kappa_i}}}_{=: f_i(s)} \times \underbrace{\left( \frac{e^{\tau s}}{s - 1/\lambda_i} \times I_1 \right)}_{=: g_i(s)}. \end{aligned}$$

Now, the residue of  $F(s)$  at  $s = -1/\lambda_i$  is transformed to that of  $G_i(s)$  at  $s = 0$ . The Laurent series expansion of  $f_i(s)$  and the Taylor series expansion of  $g_i(s)$  at  $s = 0$  are given respectively by

$$f_i(s) = \frac{1}{(\lambda_i s)^{\kappa_i}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i s} \right)^n \quad \text{and} \quad g_i(s) = \sum_{n=0}^{\infty} \frac{1}{n!} g_i^{(n)}(0) s^n. \quad (19)$$

By multiplying the two series and computing the coefficient of  $1/s$ , we obtain the residue of  $G_i(s)$  at  $s = 0$  as

$$\text{Res}_{s=0} G_i(s) = \frac{e^{-(\frac{\tau}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0) \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1} \quad (20)$$

for each  $i$ . Finally, substituting the residues into (18) yields (13).  $\blacksquare$

To compute (13), we need to compute  $\{\lambda_i\}$ ,  $\{\chi_i^{(j)}\}$  and the higher order derivatives of  $g_i(s)$ . The first two terms are easy to compute since they are related with the mean vector of size  $Kd$  and the covariance matrix of size  $Kd \times Kd$ . Furthermore, the higher order derivatives of  $g_i(s)$  can also be computed efficiently based on recursion. (Please see Appendix C-A.) Note that in the case that the elements  $\mathbf{H}_{ki}^{(w)}$  in (3) have difference variances, (13) is still valid since the difference variances only change the covariance matrix (10) and the outage expression depends on the covariance matrix (10) through  $\{\lambda_i\}$  and  $\{\chi_i^{(j)}\}$ .

Next, we provide some useful corollaries to Theorem 1 regarding the outage probability in meaningful special cases. First, we consider the case in which a subset of channels are perfectly known at receiver  $k$ , i.e.,  $\mathbf{H}_{ki} = \hat{\mathbf{H}}_{ki}$  and  $\mathbf{E}_{ki} = \mathbf{0}$  for some  $i \in \mathcal{K}$ . This corresponds to the case in which channel estimation

or CSI feedback for some links is easier than that for other links. For example, the desired link channel may be easier to estimate than others. The outage probability in this case is given by the following corollary.

*Corollary 1:* When perfect CSI for some channel links including the desired link is available at receiver  $k$ , i.e.,  $\hat{\mathbf{H}}_{ki} = \mathbf{H}_{ki}$  for  $i \in \Upsilon_k \subset \mathcal{K}$ , the outage probability for stream  $m$  of user  $k$  is given by

$$\begin{aligned} \Pr\{\text{outage}\} &= \Pr\{\log_2(1 + \text{SINR}_k^{(m)}) \leq R_k^{(m)}\} \\ &= -\sum_{i=1}^{\kappa'} \frac{e^{-(\frac{\tau'}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_{1,i}^{(n)}(0) \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1} \end{aligned} \quad (21)$$

where  $\tau'$  is defined below;  $\{\lambda_i, i = 1, \dots, \kappa'\}$  is the set of all the distinct eigenvalues of the covariance matrix (10);  $\kappa_i$  is the multiplicity of  $\lambda_i$ , satisfying  $(K - |\Upsilon_k|)d = \sum_{i=1}^{\kappa'} \kappa_i$ ;  $\chi_i^{(j)}$  is given in (14); and

$$g_{1,i}(s) = \frac{e^{\tau' s}}{s - 1/\lambda_i} \cdot \frac{\exp\left(-\sum_{p \neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2\right)}{\prod_{p \neq i} \left(1 + (s-1/\lambda_i)\lambda_p\right)^{\kappa_p}}. \quad (22)$$

*Proof:* When CSI for some links including the desired link is perfect, the outage event at stream  $m$  of user  $k$  is given by

$$\log_2 \left( 1 + \frac{|\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{kk} \mathbf{v}_k^{(m)}|^2}{\sum_{i \in \Upsilon_k} \sum_{j=1, j \neq m}^d |\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{ki} \mathbf{v}_i^{(j)}|^2 + \sum_{i \in \Upsilon_k, i \neq k} |\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{ki} \mathbf{v}_i^{(m)}|^2 + \sum_{i \in \Upsilon_k^c} \sum_{j=1}^d |\mathbf{u}_k^{(m)H} (\hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki}) \mathbf{v}_i^{(j)}|^2 + \sigma^2} \right) \leq R_k^{(m)}$$

since  $\mathbf{E}_{ki} = \mathbf{0}$  for  $i \in \Upsilon_k$ . Thus, in this case the outage event is expressed in a quadratic form as follows:

$$\sum_{i \in \Upsilon_k^c} \sum_{j=1}^d X_{ki}^{(mj)H} X_{ki}^{(mj)} \geq \frac{|\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{kk} \mathbf{v}_k^{(m)}|^2}{2^{R_k^{(m)}} - 1} - \sum_{i \in \Upsilon_k} \sum_{j=1, j \neq m}^d |\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{ki} \mathbf{v}_i^{(j)}|^2 - \sum_{i \in \Upsilon_k, i \neq k} |\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{ki} \mathbf{v}_i^{(m)}|^2 - \sigma^2 =: \tau', \quad (23)$$

and we have  $X_{ki}^{(mj)} \equiv 0$  for all  $i \in \Upsilon_k$  (See (7)). The size of  $\mathbf{X}_k^{(m)}$  now reduces to  $(K - |\Upsilon_k|)d$ , and the size of the covariance matrix  $\Sigma_k^{(m)}$  is  $(K - |\Upsilon_k|)d \times (K - |\Upsilon_k|)d$ . With the new threshold  $\tau'$ , the same argument as that in Theorem 1 can be applied to yield the result. ■

Thus, when perfect CSI is available for some links, the order of the distribution is reduced under the same structure. Next, consider the specific beam design method of interference alignment and the corresponding outage probability, which can be obtained by Corollary 1 and is given in the following corollary.

*Corollary 2:* When the desired channel link is perfectly known (i.e.  $k \in \Upsilon_k$ ) and  $\{\mathbf{V}_k\}$  and  $\{\mathbf{U}_k\}$  are designed under IA based on  $\hat{\mathcal{H}}$ , the outage probability for stream  $m$  of user  $k$  is given by

$$\Pr\{\text{outage}\} = -\sum_{i=1}^{\kappa'} \frac{1}{\lambda_i^{\kappa_i}} e^{-\frac{\tau'}{\lambda_i}} \frac{1}{(\kappa_i - 1)!} g_{1,i}^{(\kappa_i-1)}(0). \quad (24)$$

*Proof:* First, express the random term in (23) as  $\sum_{i \in \Upsilon_k^c} \sum_{j=1}^d X_{ki}^{(mj)H} X_{ki}^{(mj)} = (\mathbf{X}_k^{(m)})^H \mathbf{X}_k^{(m)}$ . When the beam is designed under IA based on  $\hat{\mathcal{H}}$ , we have  $\mathbb{E}\{\mathbf{X}_k^{(m)}\} = \mathbf{0}$  since  $\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{ki} \mathbf{v}_i^{(j)} = 0$  for all  $i \in \mathcal{K} \setminus \{k\} \supset \Upsilon_k^c$ ,  $j = 1, \dots, d$ . (See (9).) Hence,  $\chi_k^{(m)} = \mathbf{0}$  and thus  $\chi_i^{(j)} = 0$  for all  $i$  and  $j$ . (See (14).) Then, the terms in the infinite series in (21) are zero for all  $n > \kappa_i - 1$  from the fact that  $0^0 = 1$  and  $0! = 1$ , and the result follows. ■

The outage probability for single stream communication is given in Corollary 3.

*Corollary 3:* When  $d = 1$  and all eigenvalues of  $\Sigma_k^{(m)}$  are distinct, the outage probability for user  $k$  is given by

$$\begin{aligned} \Pr\{\text{outage}\} &= \Pr\{\log_2(1 + \text{SINR}_k) \leq R_k\} \\ &= -\sum_{i=1}^K \frac{e^{-(|\chi_i|^2 + \tau/\lambda_i)}}{\lambda_i} \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \left(\frac{|\chi_i|^2}{\lambda_i}\right)^n g_i^{(n)}(0), \end{aligned} \quad (25)$$

where  $g_i(s)$  in (15) reduces to  $g_i(s) = \frac{e^{\tau s}}{s-1/\lambda_i} \cdot \frac{e^{-\sum_{p \neq i} \frac{\lambda_p(s-1/\lambda_i)}{1+\lambda_p(s-1/\lambda_i)} |\chi_p|^2}}{\prod_{p \neq i} (1+\lambda_p(s-1/\lambda_i))}$ . (Here, we have omitted the stream superscripts since the stream index is unique.)

*Proof:* Since all eigenvalues are assumed to be distinct, there are  $\kappa = K$  eigenvalues with  $\kappa_i = 1$  for all  $i$ . Substituting these into Theorem 1 yields the result. ■

Now, let us consider a simpler case for  $d = 1$  with no antenna correlation. In this case, the outage probability is given as an explicit function of the channel uncertainty level  $\sigma_h^2$ , and it is given by the following corollary to Theorem 1.

*Corollary 4:* When  $d = 1$  and there is no antenna correlation, the outage probability is given by

$$\Pr\{\text{outage}\} = -\frac{1}{(\sigma_h^2)^K} e^{-(\frac{\tau}{\sigma_h^2} + \|\chi_k\|^2)} \sum_{n=K-1}^{\infty} \frac{1}{n!} g^{(n)}(0) \frac{1}{(n-K+1)!} \left(\frac{\|\chi_k\|^2}{\sigma_h^2}\right)^{n-K+1}, \quad (26)$$

where  $\chi_k = \mathbb{E}\{\mathbf{X}_k\}/\sigma_h$  and  $g(s) = \frac{e^{\tau s}}{s-1/\sigma_h^2}$ .

*Proof:* In this case, an outage at user  $k$  occurs if and only if  $\mathbf{X}_k^H \mathbf{X}_k \geq \frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{2^{R_k-1}} - \sigma^2$ . Now, the covariance matrix  $\Sigma_k$  of  $\mathbf{X}_k$  is  $\sigma_h^2 \mathbf{I}_K$  (see (10) and (11)), and thus there is only one eigenvalue  $\sigma_h^2$  with multiplicity  $K$ . Moreover,  $\chi_k = \mathbb{E}\{\mathbf{X}_k\}/\sigma_h$  from (14) since  $\Psi_k = \mathbf{I}$  and  $\Lambda_k = \sigma_h^2 \mathbf{I}$ . By substituting these into Theorem 1, the outage probability (26) is obtained. ■

### B. The behavior analysis of the outage probability based on the Chernoff bound

The obtained exact expressions for the outage probability in the previous subsection can easily be computed numerically, and will be used for the robust beam design based on the outage probability in Section IV. Before we address the outage-based robust beam design problem, let us investigate

the behavior of the outage probability as a function of several parameters. Suppose that transmit and receive beam vectors  $\{\mathbf{v}_k^{(m)}, \mathbf{u}_k^{(m)}\}$  are designed by some known method based on  $\hat{\mathcal{H}}$ . For the given beam vectors, as seen in the obtained expressions, the outage probability is a function of other system parameters such as the known channel mean  $\{\hat{\mathbf{H}}_{ki}\}$ , the noise variance  $\sigma^2$ , the channel uncertainty level  $\sigma_h^2$ , the antenna correlation  $\Sigma_t$  and  $\Sigma_r$ , and the target rate  $R_k^{(m)}$ . Here, the dependence on  $\hat{\mathbf{H}}_{kk}$ ,  $\sigma^2$  and  $R_k^{(m)}$  is via the threshold  $\tau(\hat{\mathbf{H}}_{kk}, \sigma^2, R_k^{(m)})$ , and the dependence on  $\sigma_h^2$ ,  $\Sigma_t$ ,  $\Sigma_r$  and  $\{\hat{\mathbf{H}}_{ki}, i \neq k\}$  is via  $\chi_k^{(m)}(\Sigma_k^{(m)}(\sigma_h^2, \Sigma_t, \Sigma_r), \mathbb{E}\{\mathbf{X}_k^{(m)}\}(\hat{\mathbf{H}}_{ki}))$  and the eigenvalues of  $\Sigma_{k,i}^{(m)}(\sigma_h^2, \Sigma_t, \Sigma_r)$ . This complicated dependence structure makes it difficult to analyze the properties of the outage probability as a function of the system parameters. Thus, in this subsection we apply the Chernoff bounding technique [17] to the tractable<sup>3</sup> case of  $d = 1$  to obtain insights into the outage probability as a function of several important parameters. When  $d = 1$ , the outage event is expressed as

$$\Pr\left\{\mathbf{X}_k^H \mathbf{X}_k \geq \tau = \frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{(2^{R_k} - 1)} - \sigma^2\right\} = \Pr\left\{\sum_{i=1}^K X_{ki}^H X_{ki} \geq \tau\right\}. \quad (27)$$

Since  $\mathbf{E}_{k1}, \dots, \mathbf{E}_{kK}$  are independent and circularly-symmetric complex Gaussian random matrices,  $X_{k1}, \dots, X_{kK}$  are independent and circularly-symmetric complex Gaussian random variables. (See (7).) Thus, the term on the LHS in the second bracket in (27) is a sum of independent random variables, and the Chernoff bound can be applied to yield

$$\Pr\{\mathbf{X}_k^H \mathbf{X}_k \geq \tau\} \leq e^{-\tau s} \prod_{i=1}^K \mathbb{E}\{e^{s|X_{ki}|^2}\} \quad (28)$$

for any  $s > 0$ . The moment generating function (m.g.f.) of  $|X_{ki}|^2$  ( $X_{ki} \sim \mathcal{CN}(\mu_{ki}, \sigma_{ki}^2)$ ) is given by  $\mathbb{E}\{e^{s|X_{ki}|^2}\} = \frac{1}{1 - \sigma_{ki}^2 s} \exp\left(\frac{|\mu_{ki}|^2 s}{1 - \sigma_{ki}^2 s}\right)$  for  $s < 1/\sigma_{ki}^2$ , where  $\mu_{kk} = 0$ ,  $\mu_{ki} = \mathbf{u}_k^H \hat{\mathbf{H}}_{ki} \mathbf{v}_i$  for  $i \neq k$ , and  $\sigma_{ki}^2 = \sigma_h^2(\mathbf{u}_k^H \Sigma_r \mathbf{u}_k)(\mathbf{v}_i^H \Sigma_t \mathbf{v}_i)$ . (See (7,9,11).) Therefore, the Chernoff bound on the outage probability is given by

$$\begin{aligned} \Pr\{\mathbf{X}_k^H \mathbf{X}_k \geq \tau\} &\leq e^{-\tau s} \prod_{i=1}^K \frac{1}{1 - \sigma_{ki}^2 s} \exp\left(\frac{|\mu_{ki}|^2 s}{1 - \sigma_{ki}^2 s}\right) \\ &= \exp\left\{-\left[\tau s + \sum_{i=1}^K \log(1 - \sigma_{ki}^2 s) + \sum_{i=1}^K \frac{|\mu_{ki}|^2 s}{\sigma_{ki}^2 s - 1}\right]\right\} \end{aligned} \quad (29)$$

for  $0 < s < \min_i\{1/\sigma_{ki}^2\}$ . Now, (29) provides a tool to analyze the behavior of the outage probability as a function of several important parameters. The most desired property is the behavior of the outage

<sup>3</sup>In certain cases of  $d > 1$ , Chernoff bound can still be obtained when each element in  $\mathbf{X}_k^{(m)}$  is independent of the others. Such cases include the case that there is no antenna correlation and the transmit beam vectors are orthogonal as in the IA beam case. In this case, similar results to the case of  $d = 1$  are obtained.

probability as a function of the channel uncertainty level. This behavior is explained in the following theorem.

*Theorem 2:* When  $d = 1$ , as  $\sigma_h^2 \rightarrow 0$ , the outage probability decreases to zero, and the decay rate is given by

$$\Pr\{\text{outage}\} \leq e^{-c_1} \cdot \exp(-c_2/\sigma_h^2) \quad (30)$$

for some  $c_1$  and  $c_2 > 0$  not depending on  $\sigma_h^2$ , if the target rate  $R_k$  and the designed transmit and receive beam vectors  $\{\mathbf{v}_k, \mathbf{u}_k\}$  satisfy

$$R_k < \bar{R}_k = \log_2 \left( 1 + \frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{\sum_{i=1}^K \frac{|\mu_{ki}|^2}{1 - \frac{(\mathbf{u}_k^H \boldsymbol{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \boldsymbol{\Sigma}_t \mathbf{v}_i)}{\text{tr}(\boldsymbol{\Sigma}_r)\text{tr}(\boldsymbol{\Sigma}_t)}} + \sigma^2} \right). \quad (31)$$

*Proof:* (29) is valid for any  $s \in (0, \min_i \{1/\sigma_{ki}^2\})$ . So, let  $s = 1/\sigma_h^2 \text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)$  ( $< \min_i \{1/\sigma_{ki}^2\}$ ) since  $\|\mathbf{v}_k\| = \|\mathbf{u}_k\| = 1$  and  $\sigma_{ki}^2 = \sigma_h^2(\mathbf{u}_k^H \boldsymbol{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \boldsymbol{\Sigma}_t \mathbf{v}_i) \leq \sigma_h^2 \text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)$  for all  $i$ ). Then, the exponent in (29) is given by

$$\begin{aligned} & -\frac{\tau}{\sigma_h^2 \text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)} - \sum_{i=1}^K \log \left[ 1 - \frac{(\mathbf{u}_k^H \boldsymbol{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \boldsymbol{\Sigma}_t \mathbf{v}_i)}{\text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)} \right] - \sum_{i=1}^K \frac{|\mu_{ki}|^2}{\sigma_h^2(\mathbf{u}_k^H \boldsymbol{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \boldsymbol{\Sigma}_t \mathbf{v}_i) - \sigma_h^2 \text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)} \\ & = -\frac{1}{\sigma_h^2} \underbrace{\left\{ \frac{\tau}{\text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)} + \sum_{i=1}^K \frac{|\mu_{ki}|^2}{(\mathbf{u}_k^H \boldsymbol{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \boldsymbol{\Sigma}_t \mathbf{v}_i) - \text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)} \right\}}_{(=:c_2)} - \underbrace{\sum_{i=1}^K \log \left[ 1 - \frac{(\mathbf{u}_k^H \boldsymbol{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \boldsymbol{\Sigma}_t \mathbf{v}_i)}{\text{tr}(\boldsymbol{\Sigma}_t)\text{tr}(\boldsymbol{\Sigma}_r)} \right]}_{(=:c_1)}. \end{aligned}$$

Now, substituting  $\tau = |\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2 / (2^{R_k} - 1) - \sigma^2$  into the inequality  $c_2 > 0$  yields (31).  $\blacksquare$

Theorem 2 states that the outage probability decays to zero as the CSI quality improves, more precisely, it decays exponentially w.r.t. the inverse of channel estimation MSE (or equivalently w.r.t. the channel  $K$  factor), if the target rate is below  $\bar{R}_k$ . In the Fisherian inference framework, the inverse of estimation MSE is information. Thus, another way we can view the above is that *the outage probability decays exponentially as the Fisher information for channel state increases, if the target rate is below a certain value*. So, the outage probability due to channel uncertainty is another case in which information is the error exponent as in many other inference problems. In certain cases, the condition (31) can be simplified considerably. For example, when interference-aligning beam vectors based on  $\hat{\mathcal{H}}$  are used at the transmitters and receivers, we have  $\mu_{ki} = \mathbf{u}_k^H \hat{\mathbf{H}}_{ki} \mathbf{v}_i = 0$  for  $i \neq k$  in addition to  $\mu_{kk} = 0$ , and the condition is simplified to  $R_k < \log_2 \left( 1 + \frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{\sigma^2} \right)$ . Thus, in the case of interference alignment the outage probability can be made arbitrarily small by improving the CSI quality if the target rate is strictly less than the rate obtained by using  $\hat{\mathbf{H}}_{kk}$  as the nominal channel. Next, consider the outage behavior as

the effective SNR,  $\Gamma_{eff} := |\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2 / \sigma^2$ , increases. Since the two terms determining the effective SNR are contained only in  $\tau$ , it is straightforward to see from (29) that

$$\Pr\{\text{outage}\} \leq c_3 \exp(-c_4 \Gamma_{eff}), \quad (32)$$

for some  $c_3$  and  $c_4 = s\sigma^2/(2^{R_k} - 1) > 0$  not depending on  $\Gamma_{eff}$ . Finally, consider the case in which the target rate  $R_k$  decreases. One can expect that the outage probability decays to zero if the target rate decreases to zero. The decaying behavior in this case is given in the following theorem.

*Theorem 3:* When  $d = 1$ , as  $R_k \rightarrow 0$ , the outage probability decreases to zero, and the decay rate is given by

$$\Pr\{\text{outage}\} \leq c_6 \exp\left(-\frac{c_7}{2^{R_k} - 1}\right) = c_6 \exp\left(-\frac{c'_7}{R_k + o(R_k)}\right) \quad (33)$$

for some  $c_7, c'_7 > 0$  not depending on  $R_k$ . The last equality is when  $R_k$  is near zero.

*Proof:* Let  $s$  be any positive constant contained in an interval  $(0, 1/\max_i\{\sigma_h^2(\mathbf{u}_k^H \Sigma_r \mathbf{u}_k)(\mathbf{v}_i^H \Sigma_t \mathbf{v}_i)\})$ . Then, the exponent in (29) becomes

$$\begin{aligned} & -\tau s - \underbrace{\sum_{i=1}^K \log[1 - \sigma_h^2(\mathbf{u}_k^H \Sigma_r \mathbf{u}_k)(\mathbf{v}_i^H \Sigma_t \mathbf{v}_i)s]}_{(=:c_5)} - \sum_{i=1}^K \frac{|\mu_{ki}|^2 s}{s\sigma_h^2(\mathbf{u}_k^H \Sigma_r \mathbf{u}_k)(\mathbf{v}_i^H \Sigma_t \mathbf{v}_i) - 1} \\ & = -\left(\frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{2^{R_k} - 1} - \sigma^2\right)s - c_5 = -\frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{2^{R_k} - 1}s - c'_5. \end{aligned}$$

Hence, the Chernoff bound is given by  $\Pr\{\text{outage}\} \leq c_6 \exp\left(-\frac{s|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{2^{R_k} - 1}\right) = c_6 \exp\left(-\frac{c'_7}{R_k + o(R_k)}\right)$  for some  $c'_7 > 0$ . The last equality is when  $R_k$  is near zero. In this case, we have  $2^{R_k} - 1 = (\log 2)R_k + o(R_k)$  by Taylor's expansion.  $\blacksquare$

#### IV. OUTAGE-BASED ROBUST BEAM DESIGN

In this section, we propose an outage-based beam design algorithm based on the closed-form expressions for the outage probability derived in the previous section. Our assumption is that  $\hat{\mathcal{H}}$  is given for the beam design, as mentioned earlier. Suppose that transmit and receive beamforming matrices  $\{\mathbf{V}_k, \mathbf{U}_k\}$  are designed by using any available beam design method based on  $\hat{\mathcal{H}}$ . Based on the designed  $\{\mathbf{V}_k, \mathbf{U}_k\}$  and known  $\{\hat{\mathcal{H}}, \sigma^2\}$ , one can compute and use a nominal rate for transmission. Since  $\hat{\mathcal{H}}$  is not perfect, however, an outage may occur depending on the CSI error if the nominal rate is used for transmission. Of course, the outage probability can be made small by making the transmission rate low or by improving the CSI quality, as seen in Section III-B. However, these methods are inefficient sometimes since we may have limitations in the CSI quality or need as high rate as possible for given  $\hat{\mathcal{H}}$ . Further, in many

wireless systems the target outage probability for transmission is determined and the data transmission is performed under such an outage constraint. Thus, we here consider the beam design problem when the outage probability is given as a system parameter. In particular, we consider the following per-stream based beam design problem to maximize the sum  $\epsilon$ -outage rate for given  $\hat{\mathcal{H}}$ :

$$\begin{aligned} & \underset{\{\mathbf{v}_k^{(m)}\}, \{\mathbf{u}_k^{(m)}\}}{\text{maximize}} && \sum_{k=1}^K \sum_{m=1}^d R_k^{(m)} \end{aligned} \quad (34)$$

$$\text{subject to} \quad \Pr\{\log_2(1 + \text{SINR}_k^{(m)} |_{\hat{\mathcal{H}}}) \leq R_k^{(m)}\} \leq \epsilon \quad (35)$$

$$\|\mathbf{u}_k^{(m)}\| = \|\mathbf{v}_k^{(m)}\| = 1, \quad \forall k \in \mathcal{K}, m = 1, \dots, d, \quad (36)$$

where the  $\epsilon$ -outage rate for stream  $m$  of user  $k$  is the maximum rate satisfying (35). Like other beam design problems in MIMO interference channels, the simultaneous joint optimal design for all transmit and receive beam vectors for this problem also seems difficult. Hence, we propose an iterative approach to the above sum  $\epsilon$ -outage rate maximization problem. The proposed method is explained as follows. In the first step, we initialize  $\{\mathbf{v}_k^{(m)}\}$  and  $\{\mathbf{u}_k^{(m)}\}$  properly (here a known beam design algorithm for the MIMO interference channel can be used), and then find optimal rate-tuple  $(R_1^{(1)}, \dots, R_1^{(d)}, R_2^{(1)}, \dots, R_K^{(d)})$  that maximizes the sum for given  $\{\mathbf{v}_k^{(m)}, \mathbf{u}_k^{(m)}\}$  under the outage constraint. This step is performed based on the derived outage probability expressions in the previous section. Since designing each  $R_k^{(m)}$  does not affect others, this step can be done separately for each  $R_k^{(m)}$ . Since the outage probability for stream  $m$  of user  $k$  increases monotonically w.r.t.  $R_k^{(m)}$ , the optimal  $R_k^{(m)}$  in this step is the rate with the outage probability  $\epsilon$ . In the second step, for the obtained rate-tuple and receive beam vectors  $\{\mathbf{u}_k^{(m)}\}$  in the first step, we update the transmit beam vectors  $\{\mathbf{v}_k^{(m)}\}$  to minimize the maximum of the outage probabilities of all streams and all users. (Since the outage probabilities of all streams of all users are  $\epsilon$  at the end of the first step, this means that the outage probability decreases for all streams and all users.) Here, we apply the alternating minimization technique [28] to circumvent the difficulty in the joint transmit beam design. (The change in one transmit beam vector affects the outage probabilities of other users.) That is, we optimize one transmit beam vector while fixing all the others at a time. We iterate this procedure from the first stream of transmitter 1 to the last stream of user  $K$  until this step converge. In the third step, we design the receive beam vector  $\mathbf{u}_k^{(m)}$  to minimize the outage probability at stream  $m$  of user  $k$  with the rate-tuple determined in the first step and  $\{\mathbf{v}_k^{(m)}\}$  determined in the second step for each  $(k, m)$ . This optimization can also be performed separately for each stream of each user since the receiver filter for one stream does not affect the performance of other streams. Finally, we go back to the first step with the updated transmit and receive beam vectors (in the revisited first step, the rate for each stream



will be increased by increasing the outage probability upto to  $\epsilon$  again), and iterate the procedure until the sum  $\epsilon$ -outage rate does not change. We have summarized the sum outage rate maximizing beam design algorithm in Table I.

---

### The Proposed Algorithm

---

Input: channel state estimate  $\hat{\mathcal{H}}$  and allowed outage probability  $\epsilon$ .

0. Initialize  $\{\mathbf{v}_k^{(m)}\}$  and  $\{\mathbf{u}_k^{(m)}\}$  as sets of unit-norm vectors properly.
  1. For given  $\{\mathbf{V}_k\}$  and  $\{\mathbf{U}_k\}$ , find  $(R_1^{(1)}, \dots, R_K^{(d)})$  that maximizes  $\sum_{k=1}^K \sum_{m=1}^d R_k^{(m)}$  while the outage constraint is satisfied.
  2. Update  $\{\mathbf{V}_k = [\mathbf{v}_k^{(1)}, \dots, \mathbf{v}_k^{(d)}]\}$  for  $\{R_k^{(m)}\}$  and  $\{\mathbf{U}_k^{(m)}\}$  given from step 1.
    - For pair  $(i, j)$ , fix  $\{\mathbf{v}_k^{(m)}, k = 1, \dots, K, m = 1, \dots, d\} \setminus \{\mathbf{v}_i^{(j)}\}$  and  $\{\mathbf{U}_k\}$  and solve
 
$$\mathbf{v}_i^{(j)} = \arg \min_{\mathbf{v} \in \mathbb{C}^{N_t}} \max_{k, m} \Pr\{\text{outage}_k^{(m)}\}. \quad (37)$$
 (Here, a commercial tool such as the matlab fminimax function can be used to solve (37) together with the derived outage expression.)
      - Iterate the above step from the first stream of transmitter 1 to the last stream of transmitter  $K$  until  $\{\mathbf{V}_1, \dots, \mathbf{V}_K\}$  converges.
  3. For receiver 1 to  $K$ , obtain the receive filter  $\mathbf{u}_k^{(m)}$  that minimize the outage probability of stream  $m$  of receiver  $k$  for given  $\{\mathbf{V}_k\}$  from step 2 and given  $R_k^{(m)}$  from step 1. (Here, again a commercial tool such as the matlab fmincon function can be used together with the derived outage expression.)
  4. Go to step 1 and repeat the whole procedure until the algorithm converges.
- 

TABLE I

THE PROPOSED ALGORITHM FOR SUM  $\epsilon$ -OUTAGE RATE MAXIMIZATION WITH CHANNEL UNCERTAINTY

*Theorem 4:* The proposed beam design algorithm converges.

*Proof:* It is straightforward to see that the sum  $\epsilon$ -outage rate increases monotonically for each iteration of the three steps of the proposed algorithm. Also, the maximum sum rate is bounded by the rate with perfect CSI. Hence, the algorithm converges by the monotone convergence theorem for real sequences. ■

## V. NUMERICAL RESULTS

In this section, we provide some numerical results to validate our series derivation, to examine the outage probability as a function of several system parameters and to evaluate the performance of the

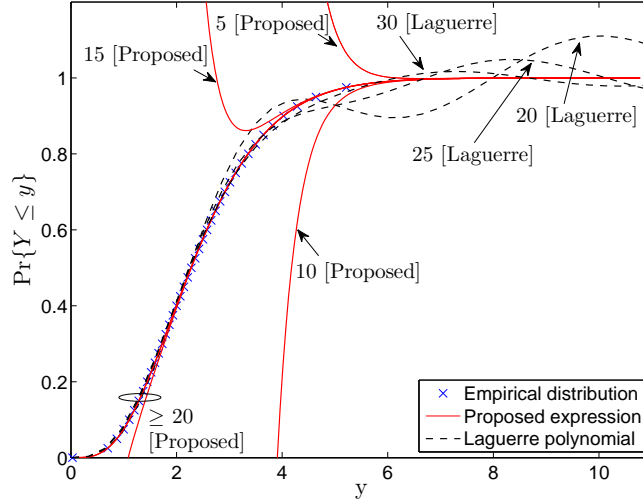


Fig. 1. Comparison of two series expressions for the CDF of quadratic form of Gaussian random variables.  $\mathbf{X} \sim \mathcal{CN}([0.5, 0.5, 0.5, 0.5]^T, 0.3\mathbf{I}_4)$ ,  $\mathbf{Q} = [1, 0.5, 0, 0; 0.5, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]$ , and  $\beta = 2$  for Laguerre series expansion.

proposed beam design algorithm. For given  $\Sigma_t$ ,  $\Sigma_r$ ,  $K_{ch}^{(ki)}$  and  $\Gamma^{(k)}$ , we first generated  $\{\hat{\mathbf{H}}_{ki}\}$  randomly according to zero-mean Gaussian distribution, and then scaled  $\hat{\mathbf{H}}_{ki}$  to yield  $\|\hat{\mathbf{H}}_{ki}\|_F^2 = N_t N_r$  for all  $(k, i)$ . In this way, the channel  $K$  factor and the SNR were simply controlled by  $\sigma_h^2$  and  $\sigma^2$ , respectively. After  $\{\hat{\mathbf{H}}_{ki}\}$  were generated as such, we generated  $\{\mathbf{E}_{ki}\}$  according to (3) and the true channel was determined by (2) if necessary<sup>4</sup>. For simplicity, we used  $K_{ch}^{(ki)} = K_{ch}$  for all  $(k, i)$  and  $\Gamma^{(k)} = \Gamma$  for all  $k$ .

First, Fig. 1 compares the convergence behavior of the derived series in this paper with that of the series fitting method [20], [21], [23], [26] based on the Laguerre basis functions for a given set of parameters shown in the label of the figure. It is seen that indeed our series converges from the upper tail first whereas the series fitting method converges from the lower tail first. (For a proof of this in the identity covariance matrix case, please refer to Appendix C-B.) Note that the series fitting method yields large error at the upper tail distribution even with a reasonably large number of terms. With this verification, next consider the outage behavior as a function of several system parameters.

<sup>4</sup>The computation of the closed-form outage probability requires only the channel statistics and  $\{\hat{\mathbf{H}}_{ki}\}$  regarding the channel information, but for Monte Carlo runs we need to generate  $\{\mathbf{E}_{ki}\}$ .

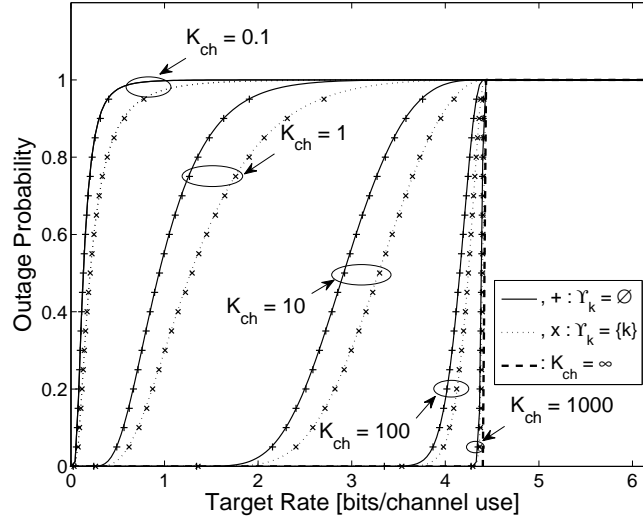


Fig. 2. Outage probability versus the target rate  $R_k$  ( $K = 3$ ,  $N_t = N_r = 2d = 2$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ ,  $\Gamma = 15$  dB. Transmit and receive beam vectors are obtained by the IIA algorithm in [3].)

Fig. 2 shows the outage probability w.r.t. the target rate  $R_k$  for a given set  $\{\hat{\mathbf{H}}_{ki}\}$  (randomly generated as above) with several different channel  $K$  factors, when  $K = 3$ ,  $N_t = N_r = 2d = 2$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ ,  $\Gamma = 15$  dB and the transmit and receive beam vectors were designed by the iterative interference alignment (IIA) algorithm [3]. The solid and dotted lines represent the result of our analysis, and the markers  $+$  and  $\times$  indicate the result of Monte Carlo runs for the outage probability. The theoretical outage curves in Fig. 2 were obtained by using (21) with the first 38 terms in the infinite series. It is seen that our analysis matches the result of Monte Carlo runs very well. The dashed line shows the outage performance when  $K_{ch} = \infty$ , i.e., all transmitters and receivers have perfect CSI. In the case of  $K_{ch} = \infty$ , we have a sharp transition behavior across  $R_{limit}$  determined by the SINR (4) with  $\mathbf{E}_{ki} = \mathbf{0}$  for all  $(k, i)$ . It is seen that the outage performance deteriorates from the ideal step curve of  $K_{ch} = \infty$ , as the CSI quality degrades. The solid lines correspond to the outage performance for the finite values of  $K_{ch}$ , when the CSI for all channel links is imperfect. It is seen that  $K_{ch} = 100$  (20 dB) yields reasonable outage performance compared with the perfect CSI case in this setup. Note that the gain in the outage probability by knowing the desired link perfectly is not negligible. (See the dotted lines.) Fig. 3 show the outage probability w.r.t. the target rate  $R_k$  for a given set  $\{\hat{\mathbf{H}}_{ki}\}$  with several different  $K_{ch}$ , when  $K = 3$ ,  $N_t = N_r = 2d = 4$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ ,  $\Gamma = 25$  dB and the transmit and receive beam vectors were designed by the IIA algorithm. Similar behavior is seen as in the single stream case, i.e., the outage performance generally deteriorates

as  $K_{ch}$  decreases. However, it is interesting to observe in the multiple stream case that sufficiently good but not perfect CSI quality yields better outage performance than the perfect CSI in the high outage probability regime. (See Fig. 3 (b).) This implies that in the multiple stream case the second term (i.e., the self inter-stream interference term) in the denominator of the SINR formula (4) is made smaller by  $\mathbf{E}_{kk}$ 's being negatively aligned with  $\mathbf{H}_{kk}$  than in the case of  $\mathbf{E}_{kk} \equiv 0$ . However, this is not useful in system operation since the system is operated in the low outage probability regime. All the theoretical curves in Figures 3 (a) and (b) were obtained by (21) with the first 45 terms in the infinite series. Fig. 4 shows the outage probability curves when the transmit and receive beamforming vectors are respectively chosen as the right and left singular vectors corresponding to the largest singular value of the desired channel and the other parameters are identical to the case in Fig. 2. A similar outage probability behavior to the previous case is observed.

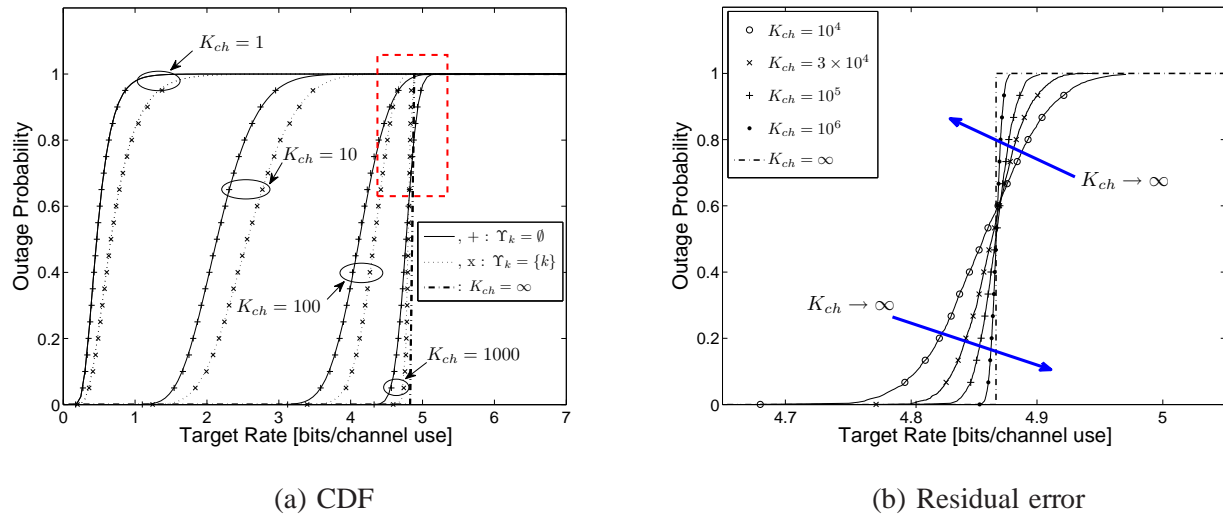


Fig. 3. Outage probability versus the target rate  $R_k$  ( $K = 3$ ,  $N_t = N_r = 2d = 4$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ ,  $\Gamma = 25$  dB. Transmit and receive beam vectors are designed by the IIA algorithm in [3].)

Next, the outage probability w.r.t. the channel  $K$  factor for a given set  $\{\hat{\mathbf{H}}\}$  for several values of the target rate  $R_k$  is shown in Fig. 5, where the outage probability along the  $y$ -axis is drawn in log scale. (The same setup as for Fig. 2 was used and the IIA algorithm is used for the transmit and receive beam design. Here, (21) with the first 38 terms in the infinite series was used to compute the analytic curves.) As predicted by Theorem 2, the outage probability indeed decays exponentially w.r.t. the channel  $K$  factor (equivalently, w.r.t. the inverse of  $\sigma_h^2$ ). The exponent depends on the target rate  $R_k$ ; the higher the target rate is, the smaller the exponent is. This decaying behavior is also predicted in Theorem 2; the

exponent  $c_2$  in (30) is proportional to  $\tau$ , and  $\tau$  is inversely proportional to the target rate  $R_k$ . It is seen that the outage probability does not decay as  $K_{ch}$  increases, if  $R_k$  is larger than  $R_{limit}$ . In addition to the exact outage probability, the Chernoff bound in this case is shown in Fig. 5 as the lines with dots and dashes. It is seen that the Chernoff bound is not very tight but the decaying slope is the same as that of the exact outage probability.

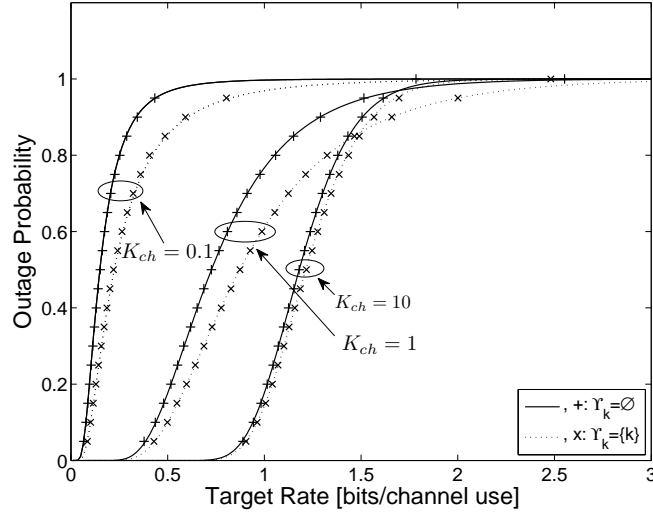


Fig. 4. Outage probability versus the target rate  $R_k$  ( $K = 3$ ,  $N_t = N_r = 2d = 2$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ ,  $\Gamma = 15$  dB. Transmit and receive beam vectors are respectively chosen as the right and left singular vectors corresponding to the largest singular value of the desired channel matrix.)

Figures 6 and 7 show the impact of antenna correlation on the outage probability. We adopted the exponential antenna correlation profile considered in [29], [30]. Under this model, the  $(i, j)$ -th element of the antenna correlation matrix  $\Sigma_t$  (or  $\Sigma_r$ ) in (3) is given by  $\rho^{|i-j|}$ , where  $\rho \in [0, 1]$  is a parameter determining the correlation strength. Since  $\text{tr}(\Sigma_t) = N_t$  and  $\text{tr}(\Sigma_r) = N_r$  for this exponential antenna correlation model, we have the same transmit and receive powers as in the case of no antenna correlation, i.e.,  $\Sigma_t = \mathbf{I}$  and  $\Sigma_r = \mathbf{I}$ . Since the outage probability depends on  $\{\hat{\mathbf{H}}_{ki}\}$  as well as on  $\Sigma_t$  and  $\Sigma_r$ , we generated one hundred  $\{\hat{\mathbf{H}}_{ki}\}$  randomly in the way that we explained already, and averaged the corresponding 100 outage probabilities to see the impact of the error correlation only. Other aspects of the system configuration were the same as those for Figures 2 and 5. It is seen that the error correlation decreases the outage probability especially when the CSI quality is very bad, but the gain becomes negligible when the CSI quality is good.

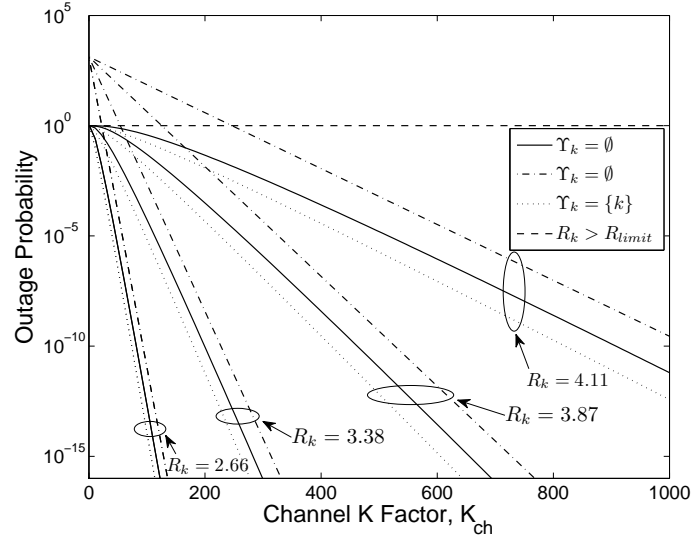


Fig. 5. Outage probability versus  $K_{ch}$  ( $K = 3$ ,  $N_t = N_r = 2d = 2$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ ,  $\Gamma = 15$  dB. Transmit and receive beam vectors are designed by the IIA algorithm in [3].)

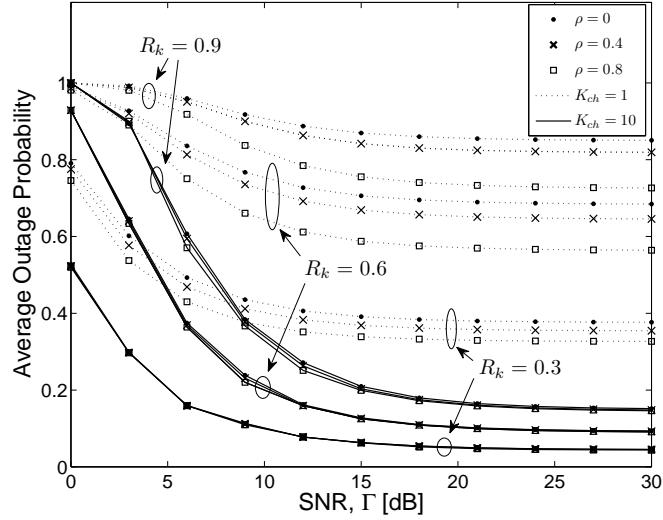


Fig. 6. Average outage probability versus  $\Gamma$  ( $K = 3$ ,  $N_t = N_r = 2d = 2$ . Transmit and receive beam vectors designed by the IIA algorithm in [3].)

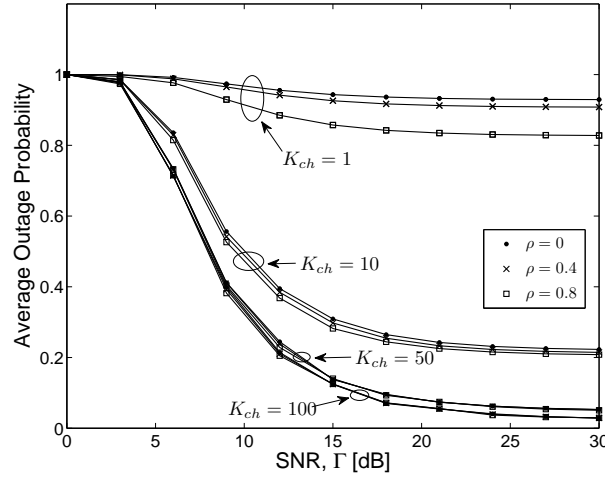


Fig. 7. Average outage probability versus  $\Gamma$  ( $K = 3$ ,  $N_t = N_r = 2d = 2$ ,  $R_k = 1.2$ . Transmit and receive beam vectors designed by the IIA algorithm in [3].)

Finally, the performance of the proposed beam design algorithm maximizing the sum  $\epsilon$ -outage rate was evaluated. As reference, we adopted the max-SINR algorithm and IIA algorithm in [3]. Although the max-SINR and IIA algorithms were originally proposed to design beam vectors with perfect channel information, we applied the algorithms to design beam vectors by treating the imperfect channel  $\hat{\mathcal{H}}$  as the true channel. The  $\epsilon$ -outage rate of the max-SINR algorithm (or the IIA algorithm) is defined as the maximum rate that can be achieved under the outage constraint of  $\epsilon$  using the beam vectors designed by the max-SINR algorithm (or the IIA algorithm). Once  $\{\mathbf{V}_k\}$  and  $\{\mathbf{U}_k\}$  are designed by any design method for given  $\Sigma_t$ ,  $\Sigma_r$  and  $\{\hat{\mathbf{H}}_{ki}\}$ , the outage probability corresponding to the designed beam vectors is easily computed as a function of the target rate  $R_k$  from Theorem 1. Thus, for the beam vectors designed by the max-SINR and IIA algorithms as well as for those designed by the proposed design algorithm in Section IV, the  $\epsilon$ -outage rate  $R_k$  can easily be obtained. Figures 8 and 9 show the sum  $\epsilon$ -outage rate of the proposed beam design method averaged over thirty different sets of  $\{\hat{\mathbf{H}}_{ki}\}$  for  $\epsilon = 0.1$  and  $\epsilon = 0.2$ , respectively, when  $K = 3$ ,  $N_t = N_r = 2d = 2$  and  $\Sigma_t = \Sigma_r = \mathbf{I}$  for different  $K_{ch}$ 's. (The outage probability expression (26) with the first 40 terms was used to compute the outage probability.) It is seen that the proposed algorithm outperforms the IIA and max-SINR algorithms in all SNR, and the max-SINR algorithm shows good performance almost comparable to the proposed algorithm at low SNR. However, as SNR increases, the performance of the max-SINR algorithm degrades to that of the IIA algorithm (the two algorithm themselves converge as SNR increases) and there is a considerable gain



by exploiting the channel uncertainty.

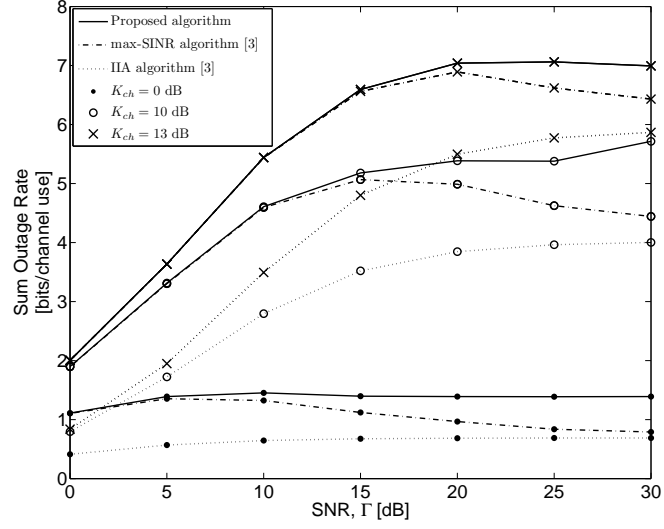


Fig. 8. Sum  $\epsilon$ -outage rate for  $\epsilon = 0.1$  ( $K = 3$ ,  $N_t = N_r = 2d = 2$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ )

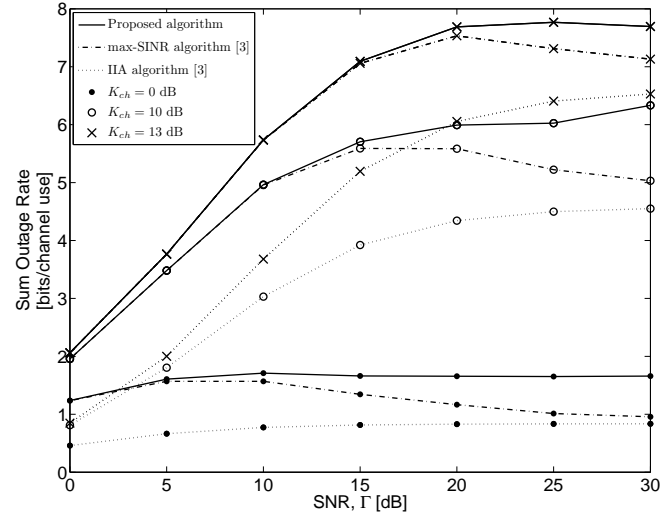


Fig. 9. Sum  $\epsilon$ -outage rate for  $\epsilon = 0.2$  ( $K = 3$ ,  $N_t = N_r = 2d = 2$ ,  $\Sigma_t = \Sigma_r = \mathbf{I}$ )

## VI. CONCLUSION

In this paper, we have considered the outage probability and the outage-based beam design for MIMO interference channels. We have derived closed-form expressions for the outage probability in MIMO interference channels under the assumption of Gaussian-distributed CSI error, and have derived the asymptotic behavior of the outage probability as a function of several system parameters based on the Chernoff bound. We have shown that the outage probability decreases exponentially w.r.t. the channel  $K$  factor defined as the ratio of the power of the known channel part and that of the unknown channel part. We have also provided an iterative beam design algorithm for maximizing the sum outage rate based on the derived outage probability expressions. Numerical results show that the proposed beam design method significantly outperforms conventional methods assuming perfect CSI in the sum outage rate performance.

## APPENDIX A

### PROOF OF (11)

The  $(p, q)$ -th element of  $\Sigma_{k,i}^{(m)}$  is given by

$$\begin{aligned}
& \mathbb{E}\{(X_{ki}^{(mp)} - \mathbb{E}\{X_{ki}^{(mp)}\})(X_{ki}^{(mq)} - \mathbb{E}\{X_{ki}^{(mq)}\})^H\} \\
&= \mathbb{E}\{(\mathbf{u}_k^{(m)H} \mathbf{E}_{ki} \mathbf{v}_i^{(p)})(\mathbf{u}_k^{(m)H} \mathbf{E}_{ki} \mathbf{v}_i^{(q)})^H\} \\
&\stackrel{(a)}{=} \mathbb{E}\{(\mathbf{v}_i^{(p)T} \otimes \mathbf{u}_k^{(m)H}) \text{vec}(\mathbf{E}_{ki}) \text{vec}(\mathbf{E}_{ki})^H (\mathbf{v}_i^{(q)T} \otimes \mathbf{u}_k^{(m)H})^H\} \\
&\stackrel{(b)}{=} \sigma_h^2 (\mathbf{v}_i^{(p)T} \otimes \mathbf{u}_k^{(m)H}) (\Sigma_t^T \otimes \Sigma_r) (\mathbf{v}_i^{(q)T} \otimes \mathbf{u}_k^{(m)H})^H \\
&\stackrel{(c)}{=} \sigma_h^2 (\mathbf{v}_i^{(p)T} \Sigma_t^T \otimes \mathbf{u}_k^{(m)H} \Sigma_r) (\mathbf{v}_i^{(q)*} \otimes \mathbf{u}_k^{(m)}), \quad \text{where } \mathbf{v}_i^{(q)*} = (\mathbf{v}_i^{(q)T})^H \\
&\stackrel{(d)}{=} \sigma_h^2 (\mathbf{v}_i^{(p)T} \Sigma_t^T \mathbf{v}_i^{(q)*} \otimes \mathbf{u}_k^{(m)H} \Sigma_r \mathbf{u}_k^{(m)}) \stackrel{(e)}{=} \sigma_h^2 (\mathbf{v}_i^{(q)H} \Sigma_t \mathbf{v}_i^{(p)}) (\mathbf{u}_k^{(m)H} \Sigma_r \mathbf{u}_k^{(m)}).
\end{aligned}$$

Here, (a) is obtained by applying  $\text{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\text{vec}(\mathbf{B})$  to each of the two terms in the expectation, (b) is by  $\mathbb{E}\{\text{vec}(\mathbf{E}_{ki}) \text{vec}(\mathbf{E}_{ki})^H\} = \sigma_h^2 (\Sigma_t^T \otimes \Sigma_r)$ , (c) and (d) are by  $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC} \otimes \mathbf{BD})$ , and finally (e) is because  $\mathbf{v}_i^{(p)T} \Sigma_t^T \mathbf{v}_i^{(q)*}$  and  $\mathbf{u}_k^{(m)H} \Sigma_r \mathbf{u}_k^{(m)}$  are scalars. ■

## APPENDIX B

### DISTRIBUTION OF A NON-CENTRAL GAUSSIAN QUADRATIC FORM

The contents in Appendices B and C are from the technical report WISRL-2012-APR-1, KAIST, "A Study on the Series Expansion of Gaussian Quadratic Forms".

### A. Previous work and literature survey

There exist extensive literature about the probability distribution and statistical properties of a quadratic form of non-central (complex) Gaussian random variables in the communications area and the probability and statistics community. Through a literature survey, we found that the main technique to compute the distribution of a central (or a non-central) Gaussian quadratic form is based on series fitting, which was concretely unified and developed by S. Kotz [20], [21], and most of other works are its variants, e.g., [23]. First, we briefly explain this series fitting method here.

Consider a Gaussian quadratic form  $\mathbf{x}^H \bar{\mathbf{Q}} \mathbf{x}$ , where  $\mathbf{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  with size  $n$  and  $\bar{\mathbf{Q}} = \bar{\mathbf{Q}}^H$ . The first step of the series fitting method is to convert the non-central Gaussian quadratic form into a linear combination of chi-square random variables:

$$\mathbf{x}^H \bar{\mathbf{Q}} \mathbf{x} = \sum_{i=1}^n \lambda_i |z_i + \delta_i|^2 = \sum_{i=1}^n \lambda_i [\text{Re}(z_i + \delta_i)^2 + \text{Im}(z_i + \delta_i)^2], \quad (38)$$

where  $z_i \stackrel{\text{independent}}{\sim} \mathcal{CN}(0, 2)$  for  $i = 1, \dots, n$ , and  $\{\delta_i, \lambda_i\}$  are constants determined by  $\bar{\mathbf{Q}}$ ,  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$ . Note that  $\text{Re}(z_i) \sim \mathcal{N}(0, 1)$  and  $\text{Im}(z_i) \sim \mathcal{N}(0, 1)$ . Thus, the non-central Gaussian quadratic form is equivalent to a weighted sum of non-central Chi-square random variables of which moment generating function (MGF) is known. The MGF of a weighted sum of  $n$  independent non-central  $\chi^2$  random variables with degrees of freedom  $2m_i$  and non-centrality parameter  $\mu_i^2$  is given by

$$\Phi(s) = \exp \left\{ -\frac{1}{2} \sum_{i=1}^n \mu_i^2 + \frac{1}{2} \sum_{i=1}^n \frac{\mu_i^2}{1 - 2\lambda_i s} \right\} \cdot \prod_{i=1}^n \frac{1}{(1 - 2\lambda_i s)^{m_i}}. \quad (39)$$

Note here that  $\Phi(-s)$  is nothing but the Laplace transform of the *probability density function (PDF)* of  $\mathbf{x}^H \bar{\mathbf{Q}} \mathbf{x}$  or equivalently  $\sum_{i=1}^n \lambda_i |z_i + \delta_i|^2$ . Now, the series fitting method expresses the PDF as an infinite series composed of a set of known basis functions and tries to find the linear combination coefficients so that the Laplace transform of this series is the same as the known  $\Phi(-s)$ . Specifically, let the PDF be

$$g_n(\bar{\mathbf{Q}}, \boldsymbol{\mu}, \boldsymbol{\Sigma}; y) = \sum_{k=0}^{\infty} c_k h_k(y), \quad (40)$$

where  $\{h_k(y), k = 0, 1, \dots\}$  is the set of known basis functions and  $\{c_k, k = 0, 1, \dots\}$  is the set of linear combination coefficients to be determined. Here, to make the problem tractable, in most cases, the following conditions are imposed. First, the sequence  $\{h_k(y)\}$  of basis functions is chosen among measurable complex-valued functions on  $[0, \infty]$  such that

$$\sum_{k=0}^{\infty} |c_k| |h_k(y)| \leq A e^{by}, \quad y \in [0, \infty] \text{ almost everywhere,} \quad (41)$$

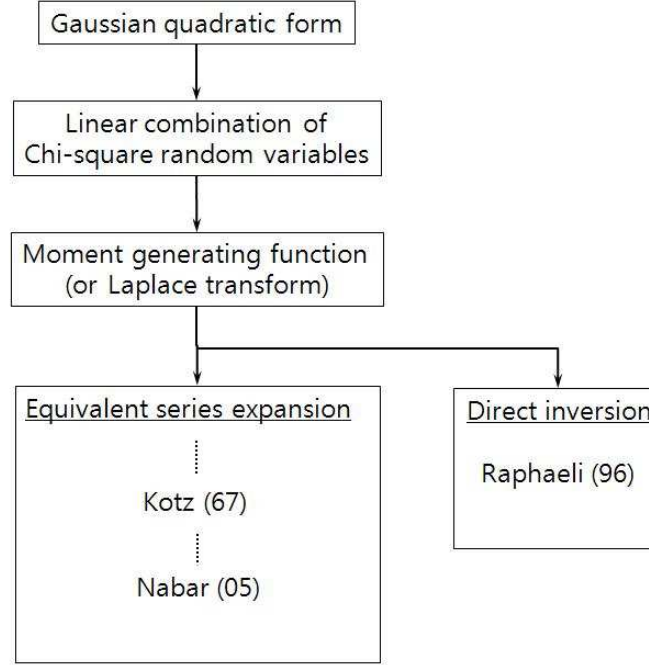


Fig. 10. Computation of the distribution of a Gaussian quadratic form

where  $A$  and  $b$  are real constants. Second, the Laplace transform  $\hat{h}_k(s)$  of  $h_k(y)$  has a special form:

$$\hat{h}_k(s) = \xi(s)\eta^k(s), \quad (42)$$

where  $\xi(s)$  is a non-vanishing, analytic function for  $\text{Re}(s) > b$ , and  $\eta(s)$  is analytic for  $\text{Re}(s) > b$  and has an inverse function. The first condition is for the existence of Laplace transform and the second condition is to make the problem tractable. Finally, with the pre-determined  $\{h_k(y)\}$  with the conditions, the coefficients  $\{c_k\}$  are computed so that

$$\mathcal{L}(g_n(\bar{\mathbf{Q}}, \boldsymbol{\mu}, \boldsymbol{\Sigma}; y)) = \sum_{k=0}^{\infty} c_k \hat{h}_k(s) = \Phi(-s), \quad (43)$$

where  $\mathcal{L}(\cdot)$  denote the Laplace transform of a function.

Widely used  $\{h_k(y)\}$  for the series expansion of the PDF of a quadratic form of non-central Gaussian random variables is as follows [20], [21].

1. (Power series):  $h_k(y) = (-1)^k \frac{(y/2)^{n/2+k-1}}{2\Gamma(n/2+k)}.$
2. (Laguerre polynomials):

$$h_k(y) = g(n; y/\beta) [k! \frac{\Gamma(n/2)}{\beta \Gamma(n/2+k)}] L_k^{(n/2-1)}(y/2\beta), \quad (44)$$

where  $g(n; y)$  is the central  $\chi^2$  density with  $n$  degrees of freedom and  $L_k^{(n/2-1)}(x)$  is the generalized Laguerre polynomial defined by Rodrigues' formula

$$L_k^{(n/2-1)}(x) = \frac{1}{k!} e^x x^{-(n/2-1)} \frac{d^k}{dx^k} e^{-x} x^{k+1}$$

for  $a > 1$  and a positive control parameter  $\beta$ .

For the detail computation of  $\{c_k\}$ , please refer to [20], [21], [26]. The whole procedure is summarized in Fig. 10.

### Reference group 1

[Kotz-67a] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables. I. Central Case," *Ann. Math. Statist.*, vol. 38, pp. 823 – 837, Jun. 1967.

[Kotz-67b] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables. II. Non-central Case," *Ann. Math. Statist.*, vol. 38, pp. 838 – 848, Jun. 1967.

[Mathai-92] A. M. Mathai and S. B. Provost, *Quadratic forms in random variables: Theory and applications*, New York: M. Dekker, 1992.

[Nabar-05] R. Nabar, H. Bolcskei, and A. Paulraj, "Diversity and Outage Performance of Space-Time Block Coded Ricean MIMO Channels", *IEEE Trans. on Wireless Commun.*, vol. 4, no. 5, Sept. 2005.

### Reference group 2

[Pachares-55] J. Pachares, "Note on the distribution of a definite quadratic form," *Ann. Math. Statist.*, vol. 26, pp. 128 – 131, Mar. 1955.  $\Rightarrow$  Power series representation of quadratic form of central Gaussian random variables.

[Shah-61] B. K. Shah and C. G. Khatri, "Distribution of a definite quadratic form for non-central normal variates," *Ann. Math. Statist.*, vol. 32, pp. 883 – 887, Sep. 1961.  $\Rightarrow$  Power series representation of quadratic form of non-central Gaussian random variables.

[Shah-63] B. K. Shah, "Distribution of definite and of indefinite quadratic forms from a non-central normal distribution," *Ann. Math. Statist.*, vol. 34, pp. 186 – 190, Mar. 1963.  $\Rightarrow$  Extends [Gurland-55] to derive a representation of quadratic form of non-central Gaussian random vector with Laguerre polynomial. Double series of Laguerre polynomials is required.

[Gurland-55] J. Gurland, "Distribution of definite and indefinite quadratic forms," *Ann. Math. Statist.*, vol. 26, pp. 122 – 127, Jan. 1955.  $\Rightarrow$  Provides a simple representation of quadratic form of central

Gaussian random vector in Laguerre polynomial.

[Gurland-56] J. Gurland, “Quadratic forms in normally distributed random variables,” *Sankhya: The Indian Journal of Statistics* vol. 17, pp. 37 – 50, Jan. 1956.  $\Rightarrow$  CDF for the indefinite quadratic form of central random variable.

[Ruben-63] H. Ruben, “A new result on the distribution of quadratic forms,” *Ann. Math. Statist.*, vol. 34, pp. 1582 – 1584, Dec. 1963.  $\Rightarrow$  Represents the CDF of quadratic form of central and non-central Gaussian random vector with central/non-central  $\chi^2$  distribution function.

[Tiku-65] M. L. Tiku, “Laguerre series forms of non central  $\chi^2$  and  $F$  distributions,” *Biometrika*, vol. 52, pp. 415 – 427, Dec. 1965.  $\Rightarrow$  Another series representation with Laguerre polynomials.

[Davis-77] A. W. Davis, “A differential equation approach to linear combinations of independent chi-squares,” *J. of the Ame. Statist. Assoc.* vol. 72, pp. 212 – 214, Mar. 1977.  $\Rightarrow$  Provides another series representation with power series.

[Imhof-61] J. P. Imhof, “Computing the distribution of quadratic forms in normal variables,” *Biometrika* vol. 48, pp. 419 – 426, Dec. 1961.  $\Rightarrow$  Provides a numerical method of computing the distribution

[Rice-80] S. O. Rice, “Distribution of quadratic forms in normal variables - Evaluation by numerical integration,” *SIAM J. Scient. Statist. Comput.*, vol. 1, no. 4, pp. 438 – 448, 1980.  $\Rightarrow$  Another numerical method of computing distribution.

[Biyari-93] K. H. Biyari and W. C. Lindsey, “Statistical distribution of Hermitian quadratic forms in complex Gaussian variables,” *IEEE Trans. Inform. Theory*, vol. 39, pp. 1076 – 1082, Mar. 1993.  $\Rightarrow$  Series expansion of multi-variate complex Gaussian random variables. This paper deals with the case that the Hermitian matrix in the quadratic form is a special block-diagonal matrix.

### Reference group 3

[Raphaeli-96] D. Raphaeli, “Distribution of noncentral indefinite quadratic forms in complex normal variables,” *IEEE Trans. Inf. Theory*, vol. 42, pp. 1002 – 1007, May 1996.

[Al-Naffouri-09] T. Al-Naffouri and B. Hassibi, “On the distribution of indefinite quadratic forms in Gaussian random variables,” in *Proc. of IEEE Int. Symp. Inf. Theory*, (Seoul, Korea), Jun.–Jul. 2009.

### B. The difference of our work from the previous works

First, let us remind our outage event in MIMO interference channels. From equations (5), (6) and (7), we have

$$\Pr\{\text{outage}\} = \Pr \left\{ \sum_{i=1}^K \sum_{j=1}^d X_{ki}^{(mj)H} X_{ki}^{(mj)} \geq \frac{|\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{kk} \mathbf{v}_k^{(m)}|^2}{2R_k^{(m)} - 1} - \sigma^2 =: \tau \right\}, \quad (45)$$

where  $X_{ki}^{(mj)}$  is a non zero-mean Gaussian random variable. Note that the outage probability is an *upper* tail probability of the distribution of the Gaussian quadratic form  $\sum_{j=1}^d X_{ki}^{(mj)H} X_{ki}^{(mj)}$ . However, as seen in Fig. 11, *the most widely-used series fitting method explained in the previous subsection yields a good approximation of the distribution at the lower tail not at the upper tail*. The discrepancy between the series and the true PDF is large at the upper<sup>5</sup> tail for a truncated series. *On the other hand, our approach yields a good approximation to the true distribution at the upper tail*. Thus, the proposed series is more relevant to our problem than the series fitting method.

Our approach to the upper tail approximation is based on the recent works by Raphaeli [22] and by Al-Naffouri and Hassibi [25]. First, let us explain Raphaeli's method. The procedure in Fig. 10 up to obtaining the MGF of the Gaussian quadratic form is common to both the sequence fitting method and Raphaeli's method. However, Raphaeli's method obtains the PDF by direct inverse Laplace transform of the MGF  $\Phi(s)$ . Typically, the inverse Laplace transform of the MGF is represented as a complex contour integral and then the complex contour integral is computed as an infinite series by the residue theorem. However, to obtain the cumulative distribution function (CDF), which is actually necessary to compute the tail probability, Raphaeli's method requires one more step, the integration of the PDF, to obtain the CDF since the MGF  $\Phi(s)$  is the Laplace transform of the *PDF*.

To obtain the CDF of a general Gaussian quadratic form, we did not use the MGF  $\Phi(s)$ , which is a bit complicated and requires an additional step, like Raphaeli, but instead we directly used a simple contour integral for the CDF (12), obtained by Al-Naffouri and Hassibi [25].<sup>6</sup> Then, the contour integral was computed as an infinite series by the residue theorem. (Using the residue theorem is borrowed from

<sup>5</sup>In the case of the problem considered in [23], the outage defined in [23] is associated with the lower tail of the distribution and thus the series fitting method is well suited to that case. However, our system setup and considered problem are different from those in [23].

<sup>6</sup>In [25], Al-Naffouri and Hassibi obtained the contour integral, (12) for the CDF of a Gaussian quadratic form. However, they did not obtain closed-form series expressions for the contour integral in general cases except a few simple cases. The main goal of [25] was to derive a nice and simple contour integral form for the CDF.



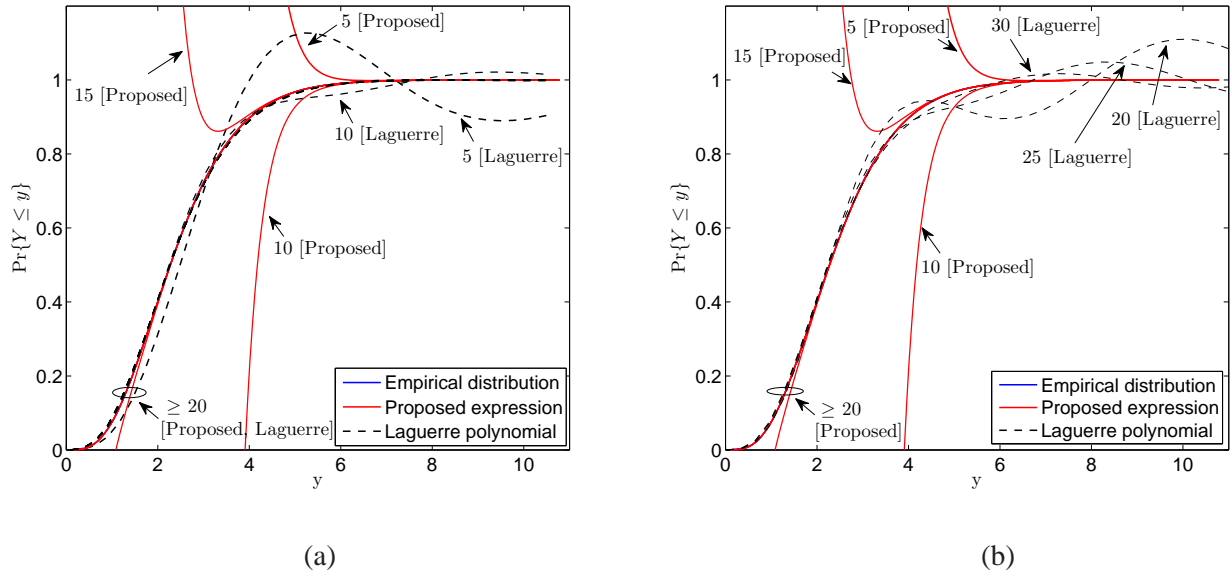


Fig. 11. Series fitting method versus direct inverse Laplace transform method: number of variables = 4,  $\mu = 0.5\mathbf{1}$ ,  $\bar{\mathbf{Q}} = [1, 0.5, 0, 0; 0.5, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]$ , and  $\Sigma = 0.3\mathbf{I}$ . (a)  $\beta = 1$  and (b)  $\beta = 2$ . ( $\beta$  is the control parameter for the Laguerre polynomials in (44).) Note that the convergent speed of the series fitting method based on the Laguerre polynomials depends much on  $\beta$ . In the case of  $\beta = 2$ , the series fitting method based on the Laguerre polynomials yields large errors at the upper tail. It is not simple how to choose  $\beta$  and an efficient method is not known. (One cannot run simulations for empirical distributions for all cases.) The series fitting method based on the power series shows bad performance, and it cannot be used in practice.

Raphaeli's work.) Thus, our result is simpler than Raphaeli's approach and does not require the integration of a PDF for the CDF.

As mentioned already, the series expansion in this paper has a particular advantage over the series fitting method considered in [23] for the outage event defined in this paper; The series in this paper fits the upper tail of the distribution well with a few number of terms. We shall provide a detailed proof for this in a special case in the next subsection. Thus, our series expressions for outage probability in MIMO interference channels are meaningful and relevant.

## APPENDIX C

## COMPUTATIONAL ISSUES AND CONVERGENCE OF THE OBTAINED SERIES

## A. Computing higher order derivatives

The general outage expression in Theorem 1 is given by

$$\begin{aligned} \Pr\{\text{outage}\} &= \Pr\{\log_2(1 + \text{SINR}_k^{(m)}) \leq R_k^{(m)}\} \\ &= -\sum_{i=1}^{\kappa} \frac{e^{-(\frac{\tau}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0) \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1} \end{aligned} \quad (46)$$

where

$$g_i(s) = \frac{e^{\tau s}}{s - 1/\lambda_i} \cdot \frac{\exp\left(-\sum_{p \neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2\right)}{\prod_{p \neq i} \left(1 + (s - 1/\lambda_i)\lambda_p\right)^{\kappa_p}}. \quad (47)$$

To compute (46), we need to compute

- $\{\lambda_i\}$  (the eigenvalues of the  $Kd \times Kd$  covariance matrix  $\mathbf{\Sigma} = \mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^H$ ),
- $\{\chi_i^{(j)}\}$  (the elements of  $Kd$  vector  $\mathbf{\chi} = \mathbf{\Lambda}^{-1/2}\mathbf{\Psi}^H\mathbf{\mu}$ , where  $\mathbf{\mu}$  is the mean vector of the Gaussian distribution),
- and the higher order derivatives of  $g_i(s)$ .

The computation of  $\{\lambda_i\}$  and  $\{\chi_i^{(j)}\}$  is simple since the sizes of the mean vector and the covariance matrix are  $Kd$  and  $Kd \times Kd$ , respectively. Furthermore, the higher order derivatives of  $g_i(s)$  can also be computed efficiently based on recursion [26], [22]. Note that  $g_i(s) = e^{\log g_i(s)}$ . Thus, the derivative of  $g_i(s)$  can be written as

$$\begin{aligned} g_i^{(1)}(s) &= g_i(s)[\log g_i(s)]^{(1)}, \\ g_i^{(2)}(s) &= g_i^{(1)}(s)[\log g_i(s)]^{(1)} + g_i(s)[\log g_i(s)]^{(2)}, \\ &\vdots \\ g_i^{(n)}(s) &= \sum_{l=0}^{n-1} \binom{n-1}{l} g_i^{(l)}(s)[\log g_i(s)]^{(n-l)}, \quad n \geq 1 \end{aligned} \quad (48)$$

where  $g_i^{(l)}(s)$  and  $[\log g_i(s)]^{(l)}$  denote the  $l$ -th derivatives of  $g_i(s)$  and  $\log g_i(s)$ , respectively. Here,  $[\log g_i(s)]^{(n)}$  can be computed from (47) as

$$[\log g_i(s)]^{(n)} = \tau \delta_{1n} - \frac{(n-1)!(-1)^{n-1}}{(s-1/\lambda_i)^n} - \sum_{p \neq i} \frac{n!(-1)^{n-1}\lambda_p^n}{(1+\lambda_p(s-1/\lambda_i))^{n+1}} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2 - \sum_{p \neq i} \frac{(n-1)!(-1)^{n-1}\kappa_p\lambda_p^n}{(1+\lambda_p(s-1/\lambda_i))^n}$$

where  $\delta_{1n}$  is Kronecker delta function. Thus, for given  $g_i(s)$  and  $[\log g_i(s)]^{(l)}$ , we can compute  $g_i^{(l)}(s)$  efficiently in a recursive way, as shown in (48).

### B. Convergence analysis

In this subsection, we provide some convergence analysis on the derived series expansion in Sec. III. Consider the general result in Theorem 1 for the CDF of a Gaussian quadratic form:

$$\Pr\{Y \leq y\} = 1 + \sum_{i=1}^{\kappa} \frac{e^{-(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0, y) \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1} \quad (49)$$

where

$$g_i(s, y) = \frac{e^{sy}}{s - \lambda_i^{-1}} \cdot \frac{\exp\left(-\sum_{p \neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2\right)}{\prod_{p \neq i} \left(1 + (s - 1/\lambda_i)\lambda_p\right)^{\kappa_p}}.$$

Here, we explicitly use the variable  $y$  as an input parameter of the function  $g_i(s)$  for later explanation.  $g_i^{(n)}(s, y)$  denotes the  $n$ -th partial derivative of  $g_i(s, y)$  with respect to  $s$ . (Here,  $\kappa$  is the number of distinct eigenvalues of the  $Kd \times Kd$  covariance matrix  $\Sigma$  and  $\kappa_i$  is the geometric order of eigenvalue  $\lambda_i$ .  $\sum_{i=1}^{\kappa} \kappa_i = Kd$ .) The residual error caused by truncating the infinite series after the first  $N$  terms is given by

$$R_N(y) = \sum_{i=1}^{\kappa} \frac{e^{-(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=N+1}^{\infty} \frac{1}{n!} g_i^{(n)}(0, y) \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1}, \quad (50)$$

and we have

$$\Pr\{Y \leq y; \text{infinite sum}\} = \Pr\{Y \leq y; \text{truncation at } N\} + R_N(y).$$

The truncation error  $R_N(y)$  can be expressed as

$$R_N(y) = \sum_{i=1}^{\kappa} R_N^i(y), \quad (51)$$

where

$$R_N^i(y) = \frac{e^{-(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=N+1}^{\infty} \frac{1}{n!} g_i^{(n)}(0, y) \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1} \quad (52)$$

for each  $1 \leq i \leq \kappa$ . Then, the magnitude of each term  $|R_N^i(y)|$  in the truncation error is bounded as

$$|R_N^i(y)| \leq \frac{1}{\lambda_i^{\kappa_i}} \exp\left\{-\left(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2\right)\right\} \cdot \sum_{n=N+1}^{\infty} \frac{1}{n!} |g_i^{(n)}(0, y)| \cdot \frac{1}{(n - \kappa_i + 1)!} \left( \frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n - \kappa_i + 1}. \quad (53)$$

As seen in Fig. 11, our series expansion fits the upper tail distribution first. Now, to assess the overall convergence speed of our series, for the same step as in Fig. 11, we ran some simulations to obtain an

empirical distribution, and computed the overall mean square error (MSE) between the truncated series and the empirical distribution over  $0 \leq y \leq 10$  as

$$\text{CDF MSE} = \frac{1}{200} \sum_{i=1}^{200} \left| \Pr\{Y \leq y_i; N, \text{type of series}\} - \Pr\{Y \leq y_i; \text{empirical}\} \right|^2,$$

where  $\{y_i\}$  are the uniform samples of  $[0, 10]$ . Fig. 12 shows the CDF MSE of the three methods in Fig. 11: the proposed series, the series fitting method with  $\beta = 1$  and the series fitting method with  $\beta = 2$ . It is seen in Fig. 12 that the overall convergence of the proposed series can be worse than

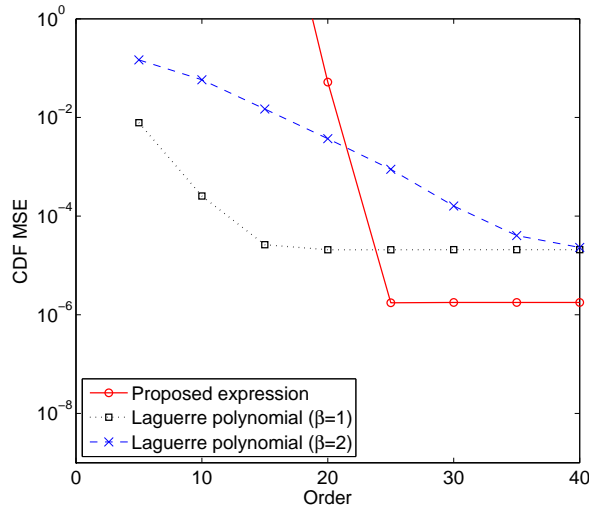


Fig. 12. CDF MSE of the CDFs in Fig. 11

the series fitting method at the small values for the number of summation terms for the setting in Fig. 11. The bad overall convergence is due to worse fitting at the lower tail of the distribution, but the bad lower tail approximation is not important to our outage computation. (Please see Fig. 11.) Fig. 13 shows another case. In this case, the proposed series outperforms the series fitting method both in the overall convergence and in the upper tail convergence. It is seen numerically that the proposed series fits the upper tail distribution first. Now, we shall prove this property of the proposed series. However, it is a difficult problem to prove this property in general cases. Thus, in the next subsection, we provide a proof of this property when the number of distinct eigenvalues of the covariance matrix  $\Sigma$  is one, e.g., in the i.i.d. case.

1) *The identity covariance matrix case:* Suppose that there is only one eigenvalue,  $\lambda (> 0)$ , with multiplicity  $\kappa$  for the covariance matrix  $\Sigma$ . This case corresponds to Corollary 4, and the outage probability

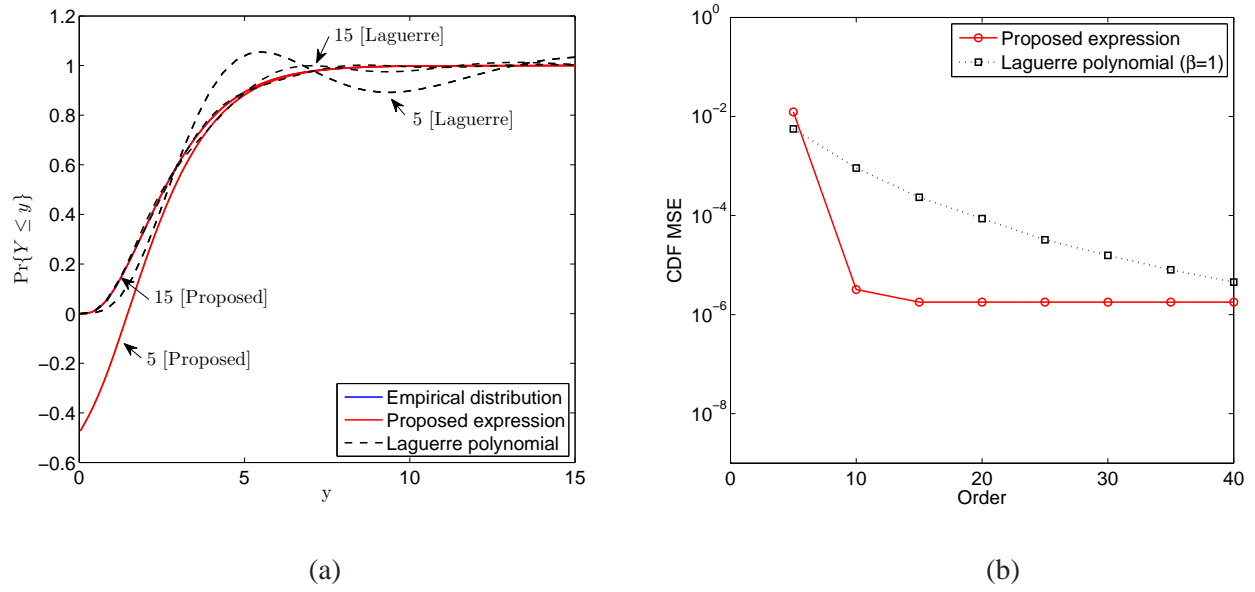


Fig. 13. number of variables = 4,  $\mu = 0.51$ ,  $\bar{\mathbf{Q}} = \mathbf{I}$ , and  $\Sigma = [0.2641 \ 0.0328 \ 0.1963 \ 0.1140; \ 0.0328 \ 0.6097 \ -0.1739 \ 0.1708; \ 0.1963 \ -0.1739 \ 0.8746 \ -0.0022; \ 0.1140 \ 0.1708 \ -0.0022 \ 0.1250]$ . In this case eigenvalues are 1.0000, 0.6318, 0.2158, and 0.0259 with  $\beta = 1$ . (a) CDF, (b) CDF MSE. Uniform sample of  $y$  is taken over  $[0, 15.9]$ .

is given by

$$\Pr\{Y \leq y\} = 1 + \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=\kappa-1}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!}, \quad (54)$$

where

$$g(s, y) = \frac{e^{ys}}{s - \lambda^{-1}} \quad (55)$$

and  $\eta^2 = \sum_{j=1}^{\kappa} |\chi^{(j)}|^2$ . The residual error caused by truncating the infinite series after the first  $N$  terms is given by

$$R_N(y) = \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!}. \quad (56)$$

Before we proceed, we first obtain the  $n$ -th derivative of  $g(s, y)$  at  $s = 0$ , which is given in the following lemma.

*Lemma 1:* For  $n \geq 0$ ,

$$g^{(n)}(0, y) = -\lambda \sum_{k=0}^n \frac{n!}{(n-k)!} \lambda^k y^{n-k}. \quad (57)$$

*Proof:* Proof is given by induction. The validity of the claim for  $n = 0, 1$  and  $2$  is shown by direction computation:

$$\begin{aligned}
g^{(0)}(0, y) &= \frac{ye^{ys}}{s - 1/\lambda} \Big|_{s=0} = -\lambda = -\lambda \sum_{k=0}^0 \frac{0!}{(0-k)!} \lambda^k y^{0-k}, \\
g^{(1)}(0, y) &= \frac{ye^{ys}(s - 1/\lambda) - e^{ys}}{(s - 1/\lambda)^2} \Big|_{s=0} = -\lambda(y + \lambda) = -\lambda \sum_{k=0}^1 \frac{1!}{(1-k)!} \lambda^k y^{1-k}, \\
g^{(2)}(0, y) &= \frac{(ye^{ys}(ys - y/\lambda - 1) + e^{ys}y)(s - \frac{1}{\lambda})^2 - 2e^{ys}(ys - y/\lambda - 1)(s - \frac{1}{\lambda})}{(s - 1/\lambda)^4} \Big|_{s=0} \\
&= -\lambda(y^2 + 2\lambda y + 2\lambda^2) = -\lambda \sum_{k=0}^2 \frac{2!}{(2-k)!} \lambda^k y^{2-k}.
\end{aligned}$$

Now, suppose that (57) holds up to the  $(n-1)$ -th derivative of  $g(s, y)$ . From the recursive formula in (48),  $g^{(n)}(0, y)$  is obtained as

$$\begin{aligned}
g^{(n)}(0, y) &= \sum_{k=0}^{n-1} \binom{n-1}{k} g^{(k)}(0, y) (\log g(0, y))^{(n-k)} \\
&= \binom{n-1}{0} g^{(0)}(0, y) (\log g(0, y))^{(n)} + \binom{n-1}{1} g^{(1)}(0, y) (\log g(0, y))^{(n-1)} + \dots \\
&\quad + \binom{n-1}{n-1} g^{(n-1)}(0, y) (\log g(0, y))^{(1)}. \tag{58}
\end{aligned}$$

Since  $[\log g(s)] = ys - \log(s - 1/\lambda)$ , we can easily see that  $[\log g(0)]^{(1)} = y + \lambda$  and  $[\log g(0)]^{(n)} = (n-1)!\lambda^n$  for  $n \geq 2$ . Therefore, (58) can be rewritten as

$$\begin{aligned}
g^{(n)}(0, y) &= (n-1)!g(0, y)\lambda^n + (n-1)g^{(1)}(0, y)(n-2)!\lambda^{n-1} + \binom{n-1}{2}g^{(2)}(0, y)(n-3)!\lambda^{n-2} + \dots \\
&\quad + (n-1)g^{(n-2)}(0, y)\lambda^2 + g^{(n-1)}(0, y)(y + \lambda) \\
&= (n-1)!g(0, y)\lambda^n + (n-1)g^{(1)}(0, y)\lambda^{n-1} + \frac{(n-1)!}{2}g^{(2)}(0, y)\lambda^{n-2} + \dots \\
&\quad + (n-1)g^{(n-2)}(0, y)\lambda^2 + \lambda g^{(n-1)}(0, y) + yg^{(n-1)}(0, y) \\
&\stackrel{(a)}{=} -\lambda \left[ \sum_{l=0}^{n-1} \frac{(n-1)!}{l!} \left( \sum_{k=0}^l \frac{l!}{(l-k)!} \lambda^k y^{l-k} \right) \lambda^{n-l} + y \sum_{m=0}^{n-1} \frac{(n-1)!}{(n-m-1)!} \lambda^m y^{n-m-1} \right] \\
&= -\lambda \left[ \sum_{l=0}^{n-1} \frac{(n-1)!}{l!} \left( \sum_{k=0}^l \frac{l!}{(l-k)!} \lambda^k y^{l-k} \right) \lambda^{n-l} + \sum_{m=0}^{n-1} \frac{(n-1)!}{(n-m-1)!} \lambda^m y^{n-m} \right] \tag{59}
\end{aligned}$$

where (a) holds since (57) holds for all  $g^{(0)}(0, y), \dots, g^{(n-1)}(0, y)$  by the induction assumption.

Here, consider the coefficient of each  $y^i$  in (59) for  $i = 0, \dots, n$ .

i)  $y^n$  is obtained only when  $m = 0$ . The coefficient of  $y^n$  from (59) is therefore given by  $-\lambda$ . It corresponds to the coefficient of  $y^n$  in (57).

ii) For  $0 < p \leq n$ , the coefficient of  $y^{n-p}$  is obtained by considering all  $(l, k)$  that satisfies  $l - k = n - p$  due to the first term in the right-hand side (RHS) of (59), and  $m = p$  due to the second term of the RHS of (59). In the first case, we obtain  $y^{n-p}$  with the following pairs  $(l, k) = (n-1, p-1), (n-2, p-2), \dots, (n-p, 0)$ . For these  $(l, k)$  pairs, we have

$$-\lambda \sum_{l=n-p}^{n-1} \frac{(n-1)!}{l!} \cdot \left( \frac{l!}{(n-p)!} \lambda^{l-n+p} y^{n-p} \right) \cdot \lambda^{n-l} = -\lambda \sum_{l=n-p}^{n-1} \frac{(n-1)!}{(n-p)!} \lambda^p y^{n-p} = -\lambda p \frac{(n-1)!}{(n-p)!} \lambda^p y^{n-p}. \quad (60)$$

In the second case of  $m = p$ , we have

$$-\lambda \frac{(n-1)!}{(n-p-1)!} \lambda^p y^{n-p}. \quad (61)$$

Finally, the coefficient of  $y^{n-q}$  is given by adding (60) and (61):

$$\begin{aligned} & -\lambda \left( \frac{(n-1)!}{(n-p-1)!} + p \frac{(n-1)!}{(n-p)!} \right) \lambda^p y^{n-p} \\ &= -\lambda \frac{(n-1)!}{(n-p-1)!} \left( 1 + \frac{p}{n-p} \right) \lambda^p y^{n-p} \\ &= -\lambda \frac{n!}{(n-p)!} \lambda^p y^{n-p}, \end{aligned}$$

which is equivalent to the coefficient for  $y^{n-p}$  in (57) ( $0 < p \leq n$ ). Thus, (57) holds for  $g^{(n)}(0, y)$ . ■

Note that  $g^{(n)}(0, y) < 0$  for all  $n \geq 0$  from (57). Therefore,  $R_N(y) \leq 0$  for all  $N$  and  $y$  and  $|g^{(n)}(0, y)| = -g^{(n)}(0, y)$ .

Now, consider the residual error term  $R_N(y)$  in (56). The magnitude of the residual error can be upper



bounded as follows:

$$\begin{aligned}
|R_N(y)| &= \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} |g^{(n)}(0, y)| \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\
&= \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} (-g^{(n)}(0, y)) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\
&= -\frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\
&= -\frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \frac{(2\eta^2)^{n-\kappa+1} (2\lambda)^{\kappa-1}}{(n-\kappa+1)!} \\
&= -(2\lambda)^{\kappa-1} \cdot \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \frac{(2\eta^2)^{n-\kappa+1}}{(n-\kappa+1)!} \\
&\stackrel{(a)}{\leq} -(2\lambda)^{\kappa-1} \cdot \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \exp(2\eta^2) \\
&= -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \\
&\stackrel{(b)}{\leq} -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \cdot \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \\
&\stackrel{(c)}{=} -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \cdot g\left(\frac{1}{2\lambda}, y\right) \\
&\stackrel{(d)}{=} -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \cdot \frac{\exp(y/2\lambda)}{-1/2\lambda} \\
&= 2^\kappa \exp(\eta^2) \cdot \exp\left(-\frac{y}{2\lambda}\right)
\end{aligned} \tag{62}$$

where (a) is from  $\frac{\gamma^k}{k!} \leq \exp(\gamma) = \sum_{p=0}^{\infty} \gamma^p/p!$  for any  $\gamma > 0$ , (b) is from the fact that summand is negative, (c) is by using the Taylor series expansion, and (d) is from (55). Since  $\eta$  is a fixed constant, from (62), for any  $N \geq 0$

$$\lim_{y \rightarrow \infty} |R_N(y)| = 0. \tag{63}$$

Thus, it is clear that the proposed series converges from the upper tail distribution!

Now, let us consider the residual error magnitude as a function of  $y$  for given  $N$ . From (57), we have

$$\frac{\partial g^{(n)}(0, y)}{\partial y} = n g^{(n-1)}(0, y). \tag{64}$$

Differentiating  $R_N(y)$  with respect to  $y$  yields

$$\begin{aligned} \frac{\partial R_N(y)}{\partial y} &= \frac{\exp(-\eta^2)}{\lambda^\kappa} \left(-\frac{1}{\lambda}\right) \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\ &\quad + \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{\partial g^{(n)}(0, y)}{\partial y} \cdot \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\ &= \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \left(-\frac{1}{\lambda} g^{(n)}(0, y) + n g^{(n-1)}(0, y)\right). \end{aligned} \quad (65)$$

Furthermore, from (57) we have

$$-\frac{1}{\lambda} g^{(n)}(0, y) + n g^{(n-1)}(0, y) = y^n. \quad (66)$$

By substituting (66) into (65), we have

$$\frac{\partial R_N(y)}{\partial y} = \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{(\eta^2/\lambda)^{n-\kappa+1} y^n}{n!(n-\kappa+1)!}, \quad (67)$$

which is positive. Since  $R_N(y) \leq 0$ ,  $\lim_{y \rightarrow \infty} R_N(y) = 0$  and  $\frac{\partial R_N(y)}{\partial y} > 0$ , the residual error magnitude monotonically decreases as  $y$  increases and the maximum error occurs at  $y = 0$  for any given  $N$ .

Now, let us compute the worst truncation error  $R_N(0)$ , which is given by

$$R_N(0) = \frac{\exp(-\eta^2)}{\lambda^\kappa} \sum_{n=N+1}^{\infty} g^{(n)}(0, 0) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!}. \quad (68)$$

From (57), we have  $g^{(n)}(0, 0) = -n! \lambda^{n+1}$ . Therefore,

$$\begin{aligned} R_N(0) &= \frac{\exp(-\eta^2)}{\lambda^\kappa} \sum_{n=N+1}^{\infty} (-n! \lambda^{n+1}) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\ &= -\frac{\exp(-\eta^2)}{\lambda^\kappa} \sum_{n=N+1}^{\infty} \lambda^{n+1} \frac{(\eta^2/\lambda)^{n-\kappa+1}}{(n-\kappa+1)!} \\ &= -\frac{\exp(-\eta^2)}{\lambda^\kappa} \sum_{n=N+1}^{\infty} \frac{(\eta^2)^{n-\kappa+1}}{(n-\kappa+1)!} \cdot \lambda^\kappa \\ &= -\exp(-\eta^2) \sum_{n=N+1}^{\infty} \frac{(\eta^2)^{n-\kappa+1}}{(n-\kappa+1)!}. \end{aligned} \quad (69)$$

From (54),  $N \geq \kappa - 2$ . For general  $N \geq \kappa - 2$ , let  $m = n - \kappa + 1$ . Then,

$$R_N(0) = -\exp(-\eta^2) \sum_{m=N-\kappa+2}^{\infty} \frac{(\eta^2)^m}{m!}.$$

Note that  $\sum_{m=N-\kappa+2}^{\infty} \frac{(\eta^2)^m}{m!}$  is the residual error of the Taylor series expansion of  $\exp(x)$  after the first  $(N - \kappa + 1)$  terms. By the Taylor theorem,

$$\sum_{m=N-\kappa+2}^{\infty} \frac{(\eta^2)^m}{m!} = \frac{(\eta^2)^{N-\kappa+2}}{(N-\kappa+2)!} \exp(\alpha \eta^2) \quad (70)$$

where some  $\alpha \in [0, 1]$ . Therefore, the worst truncation error is given by

$$|R_N(0)| = \exp\left((\alpha - 1)\eta^2\right) \times \frac{(\eta^2)^{N-\kappa+2}}{(N - \kappa + 2)!} \leq \frac{(\eta^2)^{N-\kappa+2}}{(N - \kappa + 2)!}, \quad (71)$$

where the inequality holds since  $\exp((\alpha - 1)\eta^2) \leq 1$  for  $0 \leq \alpha \leq 1$ . Furthermore, the residual error magnitude is a strictly decreasing function of  $N$  for any  $y$ ,

$$|R_N(y)| > |R_{N+1}(y)|. \quad (72)$$

This can be shown easily as follows.

$$\begin{aligned} R_N(y) &= \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n - \kappa + 1)!} \\ &= \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \left\{ \sum_{n=N+2}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n - \kappa + 1)!} + g^{(N+1)}(0, y) \frac{(\eta^2/\lambda)^{N-\kappa+2}}{(N+1)!(N - \kappa + 2)!} \right\} \\ &= R_{N+1}(y) + \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \cdot g^{(N+1)}(0, y) \frac{(\eta^2/\lambda)^{N-\kappa+2}}{(N+1)!(N - \kappa + 2)!}. \end{aligned}$$

Since  $R_N(y) < 0$  and  $g^{(N+1)}(y) < 0$  for all  $y \geq 0$  and  $N$ , we have (72). Now, based on (71) and (72), with given  $\chi_k$  and  $\sigma_h^2$ , we can compute the required number  $N$  of terms in the series to achieve the desired level of accuracy since  $\eta^2$  is known.

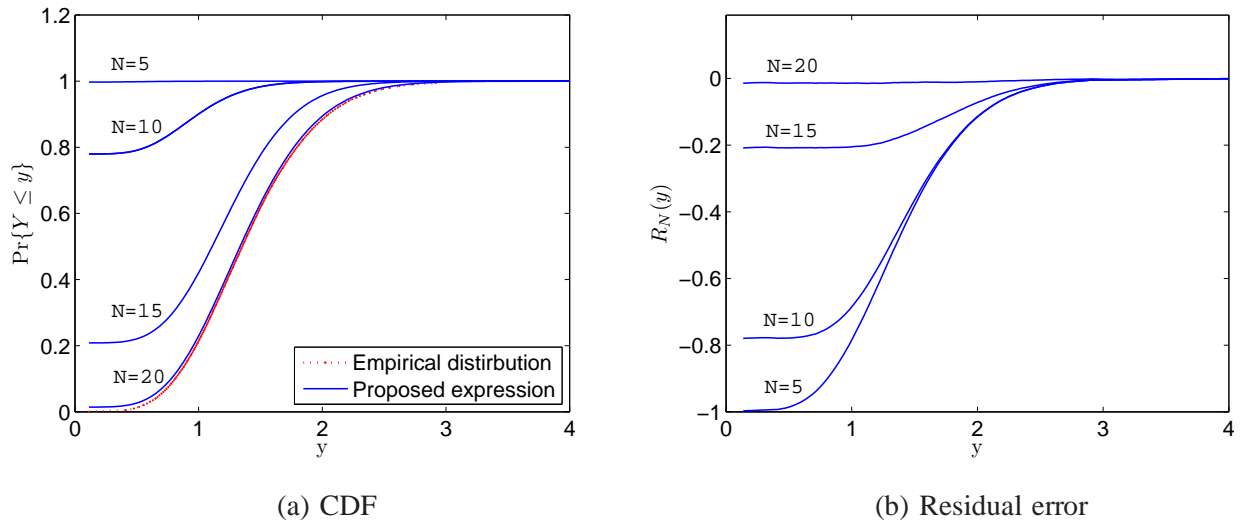


Fig. 14. number of variables = 4,  $\mu = 0.51$ ,  $\bar{\mathbf{Q}} = \mathbf{I}$ , and  $\Sigma = 0.1\mathbf{I}$ .

Finally, consider the worst case of  $N = \kappa - 2$  and  $y = 0$ :

$$R_{\kappa-2}(0) = -\exp(-\eta^2) \sum_{n=\kappa-1}^{\infty} \frac{(\eta^2)^{n-\kappa+1}}{(n - \kappa + 1)!} = -\exp(-\eta^2) \sum_{m=0}^{\infty} \frac{(\eta^2)^m}{m!} = -1,$$

where the second equality is by replacing  $m = n - \kappa + 1$ . It is easy to see that the worst case error is -1 in the identity covariance matrix case. Fig. 14 shows the performance of the proposed series expansion in the case of the identity covariance matrix. The numerical results well match our theoretical analysis in this subsection. From the figure, it seems reasonable to choose  $N \geq 20 \sim 30$  for accurate outage probability computation.

## REFERENCES

- [1] J. Park, D. Kim and Y. Sung, "Sum outage-rate maximization for MIMO interference channels," in *Proc. of IEEE Global Telecommunications Conf.*, (Houston, TX), Dec. 2011.
- [2] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom of the  $K$ -user interference channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3425 – 3441, Aug. 2008.
- [3] K. Gomadam, V. R. Cadambe, and S. A. Jafar, "A distributed numerical approach to interference alignment and applications to wireless interference networks," *IEEE Trans. Inf. Theory*, vol. 57, no. 6, pp. 3309 – 3322, Jun. 2011.
- [4] S. W. Peters and R. W. Heath, "Cooperative algorithms for MIMO interference channels," *IEEE Trans. Veh. Technol.*, vol. 60, no. 1, pp. 206 – 218, Jan. 2011.
- [5] H. Yu and Y. Sung, "Least squares approach to joint beam design for interference alignment in multiuser multi-input multi-output interference channels," *IEEE Trans. Signal Process.*, vol. 58, no. 9, pp. 4960 – 4966, Sep. 2010.
- [6] I. Santamaria, O. Gonzalez, R. W. Heath, and S. W. Peters, "Maximum sum-rate interference alignment algorithms for MIMO channels," in *Proc. of IEEE Global Telecommunications Conf.*, (Miami, FL), Dec. 2010.
- [7] D. Schmidt, S. Changxin, R. Berry, M. Honig, and W. Utschick, "Minimum mean squared error interference alignment," in *Proc. of 43rd Asilomar Conf. on Signals, Systems and Computers*, (Pacific Grove, CA), Nov. 2009.
- [8] F. Negro, S. P. Shenoy, I. Ghauri, and D. T. M. Slock, "Weighted sum rate maximization in the MIMO interference channel," in *Proc. of IEEE 21st Int. Symp. on Personal Indoor and Mobile Radio Commun.*, (Istanbul, Turkey), Sep. 2010.
- [9] J. Shin and J. Moon, "Weighted sum rate maximizing transceiver design in MIMO interference channel," in *Proc. of IEEE Global Telecommunications Conf.*, (Houston, TX), Dec. 2011.
- [10] E. Biglieri et al., *MIMO Wireless Communications*, Cambridge, UK: Cambridge University Press, 2007.
- [11] J. Lindblom, E. Karipidis, and E. Larsson, "Outage rate regions for the MISO interference channel: Definitions and interpretations," *ArXiv pre-prints cs.IT/1106.5615v1*, Jun. 2011.
- [12] J. Lindblom, E. Larsson, and E. Jorswieck, "Parametrization of the MISO IFC rate region: The case of partial channel state information," *IEEE Trans. Wireless Commun.*, vol. 9, no. 2, pp. 500–504, Feb. 2010.
- [13] W.-C. Chiang, T.-H. Chang, C. Lin, and C.-Y. Chi, "Coordinated beamforming for multiuser MISO interference channel under rate outage constraints," *ArXiv pre-prints cs.IT/1108.4475v1*, Aug. 2011.
- [14] E. Chiu, V. Lau, H. Huang, T. Wu, and S. Liu, "Robust transceiver design for  $K$ -pairs quasi-static MIMO interference channels via semi-definite relaxation," *IEEE Trans. Wireless Commun.*, vol. 9, no. 12, pp. 3762 – 3769, Dec. 2010.
- [15] H. Shen, B. Li, M. Tao, and X. Wang, "MSE-based transceiver designs for the MIMO interference channel," *IEEE Trans. Wireless Commun.*, vol. 9, no. 11, pp. 3480 – 3489, Nov. 2010.
- [16] S. Ghosh, B. Rao, and J. Zeidler, "Outage-efficient strategies for multiuser MIMO networks with channel distribution information," *IEEE Trans. Signal Process.*, vol. 58, no. 12, pp. 6312 – 6323, Dec. 2010.
- [17] H. V. Poor, *An Introduction to Signal Detection and Estimation*, 2nd ed., New York: Springer, 1994.

- [18] B. Nosrat-Makouei, J.G. Andrews, and R.W. Heath, "MIMO interference alignment over correlated channels with imperfect CSI," *IEEE Trans. Signal Process.*, vol. 59, no. 6, pp. 2783 – 2794, Jun. 2011.
- [19] J. Gurland, "Distribution of definite and of indefinite quadratic forms," *Ann. of Math. Stat.*, vol. 26, no. 1, pp. 122 – 127, Mar. 1955.
- [20] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables. I. Central case," *Ann. of Math. Stat.*, vol. 38, no. 3, pp. 823 – 837, Jun. 1967.
- [21] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables. II. Non-central case," *Ann. of Math. Stat.*, vol. 38, no. 3, pp. 838 – 848, Jun. 1967.
- [22] D. Raphaeli, "Distribution of noncentral quadratic forms in complex normal variables," *IEEE Trans. Inf. Theory*, vol. 42, no. 3, pp. 1002 – 1007, May 1996.
- [23] R. Nabar, H. Bolcskei, and A. Paulraj, "Diversity and outage performance of space-time block coded ricean MIMO channels," *IEEE Trans. Wireless Commun.*, vol. 4, no. 5, pp. 2519 – 2532, Sep. 2005.
- [24] M. O. Hasna, M. S. Alouini, and M. K. Simon, "Effect of fading correlation on the outage probability of cellular mobile radio systems," in *Proc. of IEEE Veh. Tech. Conf.*, (Atlantic City, NJ), pp. 1794 – 1998, Oct. 2001.
- [25] T. Al-Naffouri and B. Hassibi, "On the distribution of indefinite quadratic forms in Gaussian random variables," in *Proc. of IEEE Int. Symp. Inf. Theory*, (Seoul, Korea), Jun. – Jul. 2009.
- [26] A. Mathai and S. Provost, *Quadratic Forms in Random Variables: Theory and Applications*, New York: M. Dekker, 1992.
- [27] H. Sung, S. Park, K. Lee, and I. Lee, "Linear precoder designs for  $K$ -user interference channels," *IEEE Trans. Wireless Commun.*, vol. 9, no. 1, pp. 291 – 301, Jan. 2010.
- [28] I. Csiszár and P. C. Shields, *Information Theory and Statistics: A Tutorial*, Hanover, MA: Now Publishers Inc., 2004.
- [29] E.-V. Belmega, S. Lasaulce, and M. Debbah, "Power allocation games for MIMO multiple access channels with coordination," *IEEE Trans. Wireless Commun.*, vol. 8, no. 6, pp. 3182 – 3192, Jun. 2009.
- [30] A. Hjørungnes and D. Gesbert, "Precoding of orthogonal space-time block codes in arbitrarily correlated MIMO channels: Iterative and closed-form solutions," *IEEE Trans. Wireless Commun.*, vol. 6, no. 3, pp. 1072 – 1082, Mar. 2007.